

PURELY LOXODROMIC SUBGROUPS IN RIGHT-ANGLED COXETER GROUPS

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ABSTRACT. Using the work of Behrstock, Hagen, and Sisto on contact graphs for CAT(0) cubical groups, we define loxodromic elements and purely loxodromic subgroups in right-angled Coxeter groups. We prove that finitely generated purely loxodromic subgroups of a right-angled Coxeter group fulfill several equivalent conditions that parallel characterizations of subgroups with the same name in a right-angled Artin group. We compare purely loxodromic subgroups in right-angled Coxeter groups with special subgroups, star-free subgroups, and stable subgroups. We also prove several results on subgroup divergence of purely loxodromic subgroups in right-angled Coxeter groups.

1. INTRODUCTION

For each finite simplicial graph Γ the associated *right-angled Coxeter group* G_Γ has generating set S the vertices of Γ , relations $s^2 = 1$ for each s in S and relations $st = ts$ whenever s and t are adjacent vertices. In geometric group theory, groups acting on CAT(0) cube complexes are fundamental objects and right-angled Coxeter groups (RACGs) provide a rich source of these such groups. Right-angled Coxeter groups also share many beautiful properties with right-angled Artin groups (RAAGs) which are groups that have occupied an important position in geometric group theory in recent years. Moreover, the collection of RACGs is “broader” than the collection of RAAGs in the sense of coarse geometry since each right-angled Artin group can be embedded as a finite index subgroup into a right-angled Coxeter group (see [DJ00]).

The coarse geometry of RACGs was recently studied by Dani-Thomas [DT15a, DT], Dani-Stark-Thomas [DST], Behrstock-Hagen-Sisto [BHSb], Levcovitz [Lev] and others. In this article, we study a class of finitely generated subgroups of right-angled Coxeter groups, called *purely loxodromic subgroups*. These subgroups are analogous to subgroups with the same name of right-angled Artin groups (see [KK14] or [KMT]) and their infinite order elements are all loxodromic isometries of the contact graphs of associated Davis complexes (see [BHSa] for more discussion on loxoromic isometries).

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1.1. Characterization of loxodromic elements in RACGs. The *contact graph* $\mathcal{C}\Sigma$ of a CAT(0) cube complex Σ was introduced by Hagen in [Hag14]. This is the graph that encodes the hyperplane separation in Σ (see Subsection 2.3 for the precise definition). An isometry g of Σ also acts on its contact graph $\mathcal{C}\Sigma$ and g acts as a *loxodromic isometry* of the contact graph $\mathcal{C}\Sigma$ if its action has a quasi-geodesic axis in the contact graph. With a mild condition on Σ , an isometry g of Σ that acts loxodromically on $\mathcal{C}\Sigma$ can be characterized as a rank-one isometry of Σ such that no positive power of g stabilizes a hyperplane of Σ (see “Nielsen-Thurston classification” [Hag13]).

A right-angled Artin group A_Γ always acts geometrically on an associated CAT(0) cube complex, called the universal cover of its associated *Salvetti complex* \tilde{S}_Γ . When the graph Γ is connected and it does not decompose as a nontrivial join graph, a *loxodromic element* in A_Γ can be characterized by many ways. Combining results of [BC12, KK14] we state the following characterization of loxodromic element in A_Γ for easy reference to the reader.

Theorem 1.1 (Characterization of loxodromics in RAAGs, see [BC12, KK14]). *Let Γ be a finite, simplicial, connected graph which does not decompose as a nontrivial join and Γ has at least two vertices. Let g be a group element in A_Γ . Then the following are equivalent:*

- (1) *g is not conjugate into a join subgroup.*
- (2) *g acts as a rank-one isometry on \tilde{S}_Γ .*
- (3) *g acts as a loxodromic isometry of the extension graph Γ^e .*
- (4) *g is a loxodromic element (i.e. g acts as a loxodromic isometry of the contact graph $\mathcal{C}\tilde{S}_\Gamma$).*

Similar to RAAGs, each right-angled Coxeter group G_Γ also acts geometrically on an associated CAT(0) cube complex, called the *Davis complex* Σ_Γ . We also characterize *loxodromic element* in the right-angled Coxeter group G_Γ by the following theorem:

Theorem 1.2 (Characterization of loxodromics in RACGs). *Let Γ be a finite, simplicial, connected graph which does not decompose as a nontrivial join. Let g be an infinite order group element in the right-angled Coxeter group G_Γ . Then the following are equivalent:*

- (1) *g is not conjugate into a join subgroup.*
- (2) *g acts as a loxodromic isometry of the join coned-off graph $\hat{K}_j(G_\Gamma)$.*
- (3) *g acts as a loxodromic isometry of the star coned-off graph $\hat{K}_s(G_\Gamma)$.*
- (4) *g is a loxodromic element (i.e. g acts as a loxodromic isometry of the contact graph $\mathcal{C}\Sigma_\Gamma$).*

We remark that the collection of hyperplane stabilizers in a right-angled Coxeter group G_Γ is also the collection of all star subgroup conjugates in G_Γ . Therefore, there is a G_Γ -equivariant quasi-isometry between the associated star coned-off graph $\hat{K}_s(G_\Gamma)$ and the associated contact graph $\mathcal{C}\Sigma_\Gamma$ (see [Hag14]). In this paper, we also explore a G_Γ -equivariant quasi-isometry

between the associated join coned-off graph $\hat{K}_j(G_\Gamma)$ and the associated star coned-off graph $\hat{K}_s(G_\Gamma)$ (see Proposition 3.2). Therefore, loxodromic elements in RACGs can also be characterized via join coned-off graphs and star coned-off graphs as in Theorem 1.2.

We observe that loxodromic elements in both RAAGs and RACGs can be characterized algebraically by join subgroups and geometrically by their actions on associated contact graphs. However, loxodromic elements in RACGs can not be characterized by rank-one isometries of Davis complexes as in the case of RAAGs. More precisely, not all rank-one isometries of the associated Davis complex of a right-angled Coxeter group are loxodromic (see Section 3 for more detail).

1.2. Characterization of purely loxodromic subgroups in RACGs and their geometry. In [KMT], Koberda-Mangahas-Taylor study a class of finitely generated subgroups of right-angled Artin groups, called *purely loxodromic subgroups*. These are nontrivial subgroups whose nontrivial elements are all loxodromic. They also characterized purely loxodromic subgroups by using subgroup stability defined in [DT15b], subgroup action on associated contact graphs and extension graphs. More precisely, the following theorem is obtained from Theorem 1.1 and the remark after the proof of Theorem 2.4 in [KMT].

Theorem 1.3 (See [KMT]). *Let Γ be a finite, simplicial, connected graph which does not decompose as a nontrivial join. Let H be a nontrivial finitely generated subgroup of the right-angled Artin group A_Γ . Then the following are equivalent.*

- (1) *Some (any) orbit map from H into the contact graph \mathcal{CS}_Γ is a quasi-isometric embedding.*
- (2) *Some (any) orbit map from H into the extension graph Γ^e is a quasi-isometric embedding.*
- (3) *H is undistorted in A_Γ and quasi-geodesics in A_Γ between points in H have uniformly bounded Hausdorff distance from one another (i.e. H is stable in A_Γ).*
- (4) *H is purely loxodromic.*

Motivated by the work of Koberda-Mangahas-Taylor, we also study the class of finitely generated subgroups of right-angled Coxeter groups which is analogous to the class of purely loxodromic subgroups of RAAGs and we use the same name for this class of subgroups in RACGs. More precisely, a finitely generated infinite subgroup H of a right-angled Coxeter group G_Γ is *purely loxodromic* if its infinite order group elements are all loxodromic. We also characterized the such subgroup H as follows.

Theorem 1.4. *Let Γ be a simplicial connected graph which does not decompose as a nontrivial join and H a finitely generated infinite subgroup of the right-angled Coxeter group G_Γ . Then the following are equivalent:*

- (1) *Some (any) orbit map from H into the contact graph $\mathcal{C}\Sigma_\Gamma$ is a quasi-isometric embedding.*
- (2) *Some (any) orbit map from H into the star coned-off graph $\hat{K}_s(G_\Gamma)$ is a quasi-isometric embedding.*
- (3) *Some (any) orbit map from H into the join coned-off graph $\hat{K}_j(G_\Gamma)$ is a quasi-isometric embedding.*
- (4) *H is purely loxodromic.*

We can see that purely loxodromic subgroups in both RACGs and RAAGs are characterized by their quasi-isometric embedding orbit maps into associated contact graphs. Moreover, these subgroups in both RACGs and RAAGs are also characterized by the loxodromic property of their infinite order elements. Similar to purely loxodromic subgroups in RAAGs, purely loxodromic subgroups of RACGs are also *stable* (see Propositions 5.6 and 5.7). However, purely loxodromic subgroups of RACGs can not be characterized as stable subgroups as in the case of RAAGs. More precisely, not all stable subgroups of RACGs is purely loxodromic subgroup (see Subsection 3.2 for more discussion).

Here we mainly use the same strategy as in [KMT] to prove Theorem 1.4 and the stability of purely loxodromic subgroups of RACGs. More precisely, we first develop dual van Kampen diagrams for RACGs that are almost identical to dual van Kampen diagrams for RAAGs that Koberda-Mangahas-Taylor used in [KMT]. Then we use the dual van Kampen diagrams for RACGs to prove some key lemmas and propositions that are analogous to corresponding lemmas and propositions for the case of RAAGs. Reader can see that some parts of Sections 4 and 5 in this article and some parts of Sections 3, 4, 5, and 6 in [KMT] are almost identical. We also skip the proofs of some lemmas and propositions in Sections 4 and 5 in this paper because they are identical to the proofs of corresponding lemmas and propositions in [KMT].

A milder condition on infinite subgroup H of right-angled Coxeter group G_Γ is that none of its infinite order elements are conjugate into a star group. In that case, we say that H is *star-free*. For this more general class of subgroups of G_Γ , we have:

Theorem 1.5. *Let Γ be a simplicial connected graph which does not decompose as a nontrivial join and H a finitely generated infinite subgroup of the right-angled Coxeter group G_Γ . Then:*

- (1) *H is a virtually free group,*
- (2) *H is undistorted in G_Γ*

We remark that not all finitely generated subgroups of right-angled Coxeter groups is undistorted. For example, choose a right-angled Artin group A_{Γ_1} that contains a distorted subgroup H (see [Traa] or [Trac] for examples) and let G_Γ be a right-angled Coxeter group that contains A_{Γ_1} as a finite index subgroup (see [DJ00]). Then H is also a distorted subgroup

of the right-angled Coxeter group G_Γ . Theorem 1.5 indicates that, among the multiple constructions of distorted subgroups in right-angled Coxeter groups, “star-words” are a necessary common feature.

We also remark that the Theorem 1.5 is almost analogous to Theorem 1.2 in [KMT]. Statement (1) is slightly different from Statement (1) in Theorem 1.2 in [KMT] since subgroup H may contain torsion elements. However, the proof of Statement (1) of Theorem 1.5 is almost identical to the proof of Theorem 53 in [KK14] or Proposition 8.2 in [KMT]. The proof of Statement (2) is mainly used the dual van Kampen diagrams for RACGs and its proof is almost identical of the proof of Statement (2) of Theorem 1.2 in [KMT].

1.3. Connection to special subgroups and star-free subgroups. Special subgroups are typical subgroups for both RAAGs and RACGs. Roughly speaking, special subgroups in a RAAG or RACG are subgroups of the same type as the whole group (RAAG or RACG) which are induced by subgraphs of defining graph of the whole group. Special subgroups preserve many beautiful algebraic and geometric properties of the groups that contain them.

We remark that special subgroups and their conjugates in one-ended right-angled Artin groups never be purely loxodromic (even never be star-free). However, this fact is no longer true for RACGs. We study connections among purely loxodromic subgroups, star-free subgroups, special subgroups and their conjugates in RACGs in the following proposition.

Proposition 1.6. *Let Γ be a simplicial connected graph which does not decompose as a nontrivial join. Let H be a conjugate of a special subgroup induced by a subset S_1 of vertex set of Γ . Then the following are equivalent:*

- (1) S_1 contains at least two non-adjacent vertices and the distance in Γ between any two elements of S_1 is different from 2.
- (2) H is purely loxodromic.
- (3) H is star-free.

We remark that any infinite subgroup of a purely loxodromic subgroup is also purely loxodromic. Therefore, we can conclude that any infinite subgroup which is conjugate into a special subgroup with the generating set satisfying conditions in the above proposition is a purely loxodromic subgroup. In other words, Proposition 1.6 allows us to produce many non-virtually cyclic purely loxodromic subgroups in RACGs. However, not all purely loxodromic subgroups in RACGs are conjugate into purely loxodromic special subgroups (see Subsection 3.2 for example).

Proposition 1.6 also shows the equivalence between being purely loxodromic and being star-free on special subgroups and their conjugates. In general, a purely loxodromic subgroup of G_Γ is star-free, but the converse is false (see Subsection 3.2 for counter example).

1.4. Subgroup divergence of purely loxodromic subgroups in RACGs.

Subgroup divergence was introduced by the author with the name lower relative divergence in [Tra15] to study geometric embedding properties of a

subgroup inside a finitely generated group. Roughly speaking, subgroup divergence measures the distance distortion of the complement of the r -neighborhood of a subgroup in the whole group when r increases (see Subsection 2.1 for the precise definition). In geometric group theory, the famous concept subgroup distortion measures distance distortion between metric inside a subgroup and metric induced from the whole group. Subgroup divergence, in contrast to subgroup distortion, measures the distance distortion of the complement of a subgroup inside the whole group.

We know that all purely loxodromic subgroups in RACGs share many common geometric properties. They are all stable and in particular they are all undistorted. Also all non-virtually cyclic purely loxodromic subgroups in RACGs are quasi-isometric since they are all virtually free subgroups. Therefore, we need a tool to classify non-virtually cyclic purely loxodromic subgroups in RACGs. The following theorem shows that subgroup divergence can be a good tool to classify these such subgroups in term of their geometric embedding properties.

Theorem 1.7. *For each $d \geq 2$ there is a right-angled Coxeter group G_d such that for each $2 \leq m \leq d$ the group G_d contains a purely loxodromic subgroup H_d^m which is isomorphic to the group $F = \langle a, b, c \mid a^2 = b^2 = c^2 = e \rangle$ and whose subgroup divergence in G_d is a polynomial of degree m .*

In general, we remark that the image of a virtually cyclic subgroup K in a Cayley graph of the whole group is quasi-isometric to an infinite path α of the graph. Therefore, the subgroup divergence of K can be computed easily by using the “divergence” of α in the Cayley graph. Thus, the subgroup divergence of virtually cyclic purely loxodromic subgroups of group G_d in Theorem 1.7 can be computed without much difficulty. However, all purely loxodromic subgroups H_d^m in Theorem 1.7 are isomorphic to a non-virtually cyclic subgroups and we know that the image of a non-virtually cyclic subgroup in a Cayley graph of the whole group is complicated. Therefore, it is more challenging to compute the subgroup divergence of purely loxodromic subgroups H_d^m in Theorem 1.7. Here we used some results related to divergence of right-angled Coxeter groups in Dani-Thomas [DT15a] and Levcovitz [Lev] for the proof of Theorem 1.7. However, more additional techniques were required to construct purely loxodromic subgroups with desired subgroup divergence.

We also remark that Theorem 1.7 provides us first examples of non-virtually cyclic subgroups with polynomial subgroup divergence other than linear and quadratic. The reader can see the first example of non-virtually cyclic subgroups with quadratic subgroup divergence constructed by the author in [Trab]. In that paper, the author proved that the subgroup divergence of all purely loxodromic subgroups in right-angled Artin groups are exactly quadratic. Therefore, Theorem 1.7 also points out the difference on the geometry of purely loxodromic subgroups in RAAGs and in RACGs.

We also prove the quadratic lower bound on subgroup divergence of purely loxodromic subgroups in RACGs.

Theorem 1.8. *Let Γ be a simplicial, finite, connected graph with the vertex set S such that Γ does not decompose as a nontrivial join. Let H be a finitely generated purely loxodromic subgroup of the right-angled Coxeter group G_Γ . Then the subgroup divergence of H in G_Γ is at least quadratic.*

We remark that not all subgroups in right-angled Coxeter groups has super linear subgroup divergence. For example, let G_Γ be a right-angled Coxeter group that contains a finite index right-angled Artin subgroup A_{Γ_1} and let H an infinite index nontrivial normal subgroup of A_{Γ_1} (H is not necessarily finitely generated). Then the subgroup divergence of H in G_Γ is linear by Theorem 4.15 and Theorem 7.8 in [Tra15]. We also remark that subgroup divergence of star-free subgroups are not super linear in general (see the star-free subgroup which is not purely loxodromic in Subsection 3.2 for an example).

We give some criterion on defining graph Γ so subgroup divergence of all purely loxodromic subgroups in the right-angled Coxeter group G_Γ are exactly quadratic. Therefore, the quadratic lower bound in Theorem 1.8 is tight.

Theorem 1.9. *Let Γ be a simplicial, finite, connected, \mathcal{CFS} graph with the vertex set S such that Γ does not decompose as a nontrivial join. Let H be a finitely generated purely loxodromic subgroup of the right-angled Coxeter group G_Γ . Then the subgroup divergence of H in G_Γ is exactly quadratic.*

We remark that the \mathcal{CFS} condition on defining graphs was used in Dani-Thomas [DT15a] and Levcovitz [Lev] to characterize right-angled Coxeter groups with quadratic divergence. The above theorem shows that this condition is also useful to study geometric embedding properties of purely loxodromic subgroups in RACGs.

2. PRELIMINARIES

2.1. Coarse geometry. We first review the concepts of quasi-isometric embedding, quasi-isometry, quasi-geodesics, geodesics, undistorted subgroups, stable subgroups, and subgroup divergence.

Definition 2.1. For metric spaces (X, d_X) and (Y, d_Y) be two metric spaces and constants $K \geq 1$ and $L \geq 0$, a map $f : X \rightarrow Y$ is a (K, L) -quasi-isometric embedding if for all $x_1, x_2 \in X$,

$$\frac{1}{K}d_X(x_1, x_2) - L \leq d_Y(f(x_1), f(x_2)) \leq Kd_X(x_1, x_2) + L.$$

A quasi-isometric embedding is simply a (K, L) -quasi-isometric embedding for some K, L . When a quasi-isometric embedding $f : X \rightarrow Y$ has the additional property that every point in Y is within a bounded distance

from the image $f(X)$, we say f is a *quasi-isometry* and X and Y are *quasi-isometric*.

Where X is a subinterval I of \mathbb{R} or \mathbb{Z} , we call a (K, L) -quasi-isometric embedding $f: I \rightarrow Y$ a (K, L) -*quasi-geodesic*. If $K = 1$ and $L = 0$, then $f: I \rightarrow Y$ is a *geodesic*.

Definition 2.2. Let G be a finitely generated group and H a finitely generated subgroup of G . We say H is *undistorted* in G if the inclusion map of subgroup H into the group G is a quasi-isometric embedding (this is independent of the word metrics on H and G). We say H is *stable* in G if H is undistorted in G , and for any $K \geq 1$ and $L \geq 0$ there is an $M = M(K, L) \geq 0$ such that any pair of (K, L) -quasi-geodesics in G with common endpoints in H have Hausdorff distance no greater than M .

Before we define the concepts of subgroup divergence, we need to build the notions of domination and equivalence which are the tools to measure the subgroup divergence.

Definition 2.3. Let \mathcal{M} be the collection of all functions from $[0, \infty)$ to $[0, \infty]$. Let f and g be arbitrary elements of \mathcal{M} . The function f is *dominated by the function g* , denoted $f \preceq g$, if there are positive constants A, B, C and D such that $f(x) \leq Ag(Bx) + Cx$ for all $x > D$. Two function f and g are *equivalent*, denoted $f \sim g$, if $f \preceq g$ and $g \preceq f$.

Remark 2.4. A function f in \mathcal{M} is *linear, quadratic or exponential...* if f is respectively equivalent to any polynomial with degree one, two or any function of the form a^{bx+c} , where $a > 1, b > 0$.

Definition 2.5. Let $\{\delta_\rho^n\}$ and $\{\delta_\rho^m\}$ be two families of functions of \mathcal{M} , indexed over $\rho \in (0, 1]$ and positive integers $n \geq 2$. The family $\{\delta_\rho^n\}$ is *dominated by the family $\{\delta_\rho^m\}$* , denoted $\{\delta_\rho^n\} \preceq \{\delta_\rho^m\}$, if there exists constant $L \in (0, 1]$ and a positive integer M such that $\delta_{L\rho}^n \preceq \delta_\rho^{Mn}$. Two families $\{\delta_\rho^n\}$ and $\{\delta_\rho^m\}$ are *equivalent*, denoted $\{\delta_\rho^n\} \sim \{\delta_\rho^m\}$, if $\{\delta_\rho^n\} \preceq \{\delta_\rho^m\}$ and $\{\delta_\rho^m\} \preceq \{\delta_\rho^n\}$.

Remark 2.6. A family $\{\delta_\rho^n\}$ is dominated by (or dominates) a function f in \mathcal{M} if $\{\delta_\rho^n\}$ is dominated by (or dominates) the family $\{\delta_\rho^m\}$ where $\delta_\rho^m = f$ for all ρ and n . The equivalence between a family $\{\delta_\rho^n\}$ and a function f in \mathcal{M} can be defined similarly. Thus, a family $\{\delta_\rho^n\}$ is linear, quadratic, exponential, etc if $\{\delta_\rho^n\}$ is equivalent to the function f where f is linear, quadratic, exponential, etc.

Definition 2.7. Let X be a geodesic space and A a subspace of X . Let r be any positive number.

- (1) $N_r(A) = \{x \in X \mid d_X(x, A) < r\}$
- (2) $\partial N_r(A) = \{x \in X \mid d_X(x, A) = r\}$
- (3) $C_r(A) = X - N_r(A)$.

- (4) Let $d_{r,A}$ be the induced length metric on the complement of the r -neighborhood of A in X . If the subspace A is clear from context, we can use the notation d_r instead of using $d_{r,A}$.

Definition 2.8. Let (X, A) be a pair of geodesic spaces. For each $\rho \in (0, 1]$ and positive integer $n \geq 2$, we define a functions $\sigma_\rho^n : [0, \infty) \rightarrow [0, \infty]$ as follows:

For each positive r , if there is no pair of $x_1, x_2 \in \partial N_r(A)$ such that $d_r(x_1, x_2) < \infty$ and $d(x_1, x_2) \geq nr$, we define $\sigma_\rho^n(r) = \infty$. Otherwise, we define $\sigma_\rho^n(r) = \inf d_{\rho r}(x_1, x_2)$ where the infimum is taken over all $x_1, x_2 \in \partial N_r(A)$ such that $d_r(x_1, x_2) < \infty$ and $d(x_1, x_2) \geq nr$.

The family of functions $\{\sigma_\rho^n\}$ is the *subspace divergence* of A in X , denoted $\text{div}(X, A)$.

We now define the concept of subgroup divergence of a subgroup in a finitely generated group.

Definition 2.9. Let G be a finitely generated group and H its subgroup. We define the *subgroup divergence* of H in G , denoted $\text{div}(G, H)$, to be the subspace divergence of H in the Cayley graph $\Gamma(G, S)$ for some finite generating set S .

Remark 2.10. The concept of subgroup divergence was introduced by the author with the name lower relative divergence in [Tra15]. The subgroup divergence is a pair quasi-isometry invariant concept (see Proposition 4.9 in [Tra15]). This implies that the subgroup divergence of a subgroup on a finitely generated group does not depend on the choice of finite generating sets of the whole group.

2.2. Right-angled Coxeter groups, Davis complexes, rank-one isometries, star coned-off graphs, and join coned-off graphs.

Definition 2.11. Given a finite, simplicial graph Γ , the associated *right-angled Coxeter group* G_Γ has generating set S the vertices of Γ , and relations $s^2 = 1$ for all s in S and $st = ts$ whenever s and t are adjacent vertices.

Let S_1 be a subset of S . The subgroup of G_Γ generated by S_1 is a right-angled Coxeter group G_{Γ_1} , where Γ_1 is the induced subgraph of Γ with vertex set S_1 (i.e. Γ_1 is the union of all edges of Γ with both endpoints in S_1). The subgroup G_{Γ_1} is called a *special subgroup* of G_Γ . Any of its conjugates is called a *parabolic subgroup* of G_Γ .

A *reduced word* for a group element g in G_Γ is a minimal length word in the free group $F(S)$ representing g . It is well-known that if $w = v_1 v_2 \cdots v_p$ is not reduced, then there exists $1 \leq i < j \leq p$ such that $v_i = v_j$ and v_i is adjacent to each of the vertices v_{i+1}, \dots, v_{j-1} (the *Deletion Condition*). Moreover, if two reduced words w, w' define the same element of G_Γ , then w can be transformed into w' by a finite number of letter swapping operations (the *Transpose Condition*).

Let w be any word in the vertex generators. We say that $v \in S$ is in the *support* of w , written $v \in \text{supp}(w)$, if v occurs as a letter in w . For

$g \in G_\Gamma$ and w a reduced word representing g , we define the *support* of g , $\text{supp}(g)$, to be $\text{supp}(w)$. We define the *cyclically support* of g , $\text{csupp}(g)$, to be the intersection of all sets $\text{supp}(wgw^{-1})$, where each w is a group element in G_Γ . It follows from Transpose Condition that $\text{supp}(g)$ and $\text{csupp}(g)$ are well-defined. We say that u is *cyclically reduced* if $\text{csupp}(u) = \text{supp}(u)$. It is also well known that each $g \in G_\Gamma$ has a unique reduced expression wuw^{-1} with u cyclically reduced and therefore $\text{csupp}(g) = \text{supp}(u)$.

Definition 2.12. Let Γ_1 and Γ_2 be two graphs, the *join* of Γ_1 and Γ_2 is a graph obtained by connecting every vertex of Γ_1 to every vertex of Γ_2 by an edge.

Let J be an induced subgraph of Γ which decomposes as a nontrivial join. We call G_J a *join subgroup* of G_Γ . A reduced word w in G_Γ is called a *join word* if w represents element in some join subgroup. If β is a subword of w , we will say that β is a *join subword* of w when β is itself a join word.

For a vertex v of the graph Γ let $Lk(v)$ denote the subgraph of Γ induced by the vertices adjacent to v called the *link* of v and let $St(v)$ denote the subgraph spanned by v and $Lk(v)$ called the *star* of v . The special subgroup $G_{St(v)}$ is a *star subgroup* of G_Γ . Note that a star of a vertex is always a join, but the converse is generally not true. A reduced word w in G_Γ is called a *star word* if w represents element in some star subgroup. If β is a subword of w , we will say that β is a *star subword* of w when β is itself a star word. Note that a star word is always a join word, but the converse is generally not true.

The following lemma will contribute to the proof of Theorem 1.2.

Lemma 2.13. *Let Γ be a finite simplicial graph and g is not conjugate into a star subgroup. Then, $\text{csupp}(g^n) = \text{csupp}(g)$ for each non-zero integer n .*

Proof. Let u be a cyclically reduced element in G_Γ such that $g = wuw^{-1}$ and therefore $\text{csupp}(g) = \text{supp}(u)$. This is obvious from the Deletion and Transpose conditions that $\text{csupp}(g^n) \subset \text{csupp}(g)$ for each non-zero integer n . We assume for the contradiction that $\text{csupp}(g^{n_0}) \neq \text{csupp}(g)$ for some non-zero integer n_0 . Without loss of generality, we can assume $n_0 > 0$. Let $w = u_1u_2 \cdots u_k$ be a reduced word that represents u . Therefore, u^{n_0} can be represented by the concatenation W of n_0 blocks w as follows

$$W = w_1w_2 \cdots w_{n_0}, \text{ where } w_i = w \text{ for each } i.$$

and $g^{n_0} = wu^{n_0}w^{-1}$ obviously.

Since $\text{csupp}(g^{n_0}) \neq \text{csupp}(g)$, u is cyclically reduced, and using Deletion condition, we see that there are two elements a and b in different blocks of W that are labelled by the same vertex generator v such that v commutes with all letters between a and b in W . Assume that a is the m_1^{th} element of blocks w_i and b is the m_2^{th} element of blocks w_j ($i < j$) in W . Then v can commute with all elements u_j , where $j \geq m_1$ or $j \leq m_2$. If $m_1 \leq m_2$, then v commutes with all elements u_i of w . Therefore, u is a group element

in the star subgroup $G_{St(v)}$. This implies that g is conjugate into the star subgroup $G_{St(v)}$ which is a contradiction. If $m_1 > m_2$, u can be written as $vu'v$, where $|u'|_S = |u|_S - 2$. Therefore, u is not cyclically reduced which is a contradiction. Therefore, $csupp(g^n) = csupp(g)$ for each non-zero integer n . \square

Definition 2.14. Given a finite, simplicial graph Γ , the associated *Davis complex* Σ_Γ is a cube complex constructed as follows. For every k -clique, $T \subset \Gamma$, the special subgroup G_T is isomorphic to the direct product of k copies of Z_2 . Hence, the Cayley graph of G_T is isomorphic to the 1-skeleton of a k -cube. The Davis complex Σ_Γ has 1-skeleton the Cayley graph of G_Γ , where edges are given unit length. Additionally, for each k -clique, $T \subset \Gamma$, and coset gG_T , we glue a unit k -cube to $gG_T \subset \Sigma_\Gamma$. The Davis complex Σ_Γ is a CAT(0) space and the group G_Γ acts properly and cocompactly on the Davis complex Σ_Γ (see [Dav08]).

Definition 2.15. Let X be a CAT(0) space. A geodesic line L in X is said to have *rank-one* if it does not bound a flat half-plane. An isometry γ in $Aut(X)$ is said to have *rank-one* if it is hyperbolic and if some (and hence any) of its axes has rank-one.

Theorem 2.16 (Proposition 4.5 in [CF10]). *Let Γ be a finite simplicial graph with vertex set S and Σ_Γ the associated Davis complex. An infinite order element g in the right-angled Coxeter group G_Γ acts as a rank-one isometry of Σ_Γ if and only if g is not conjugate into a join subgroup $G_{\Gamma_1 * \Gamma_2} = G_{\Gamma_1} \times G_{\Gamma_2}$, where Γ_1 , Γ_2 , and $\Gamma_1 * \Gamma_2$ are induced subgraphs of Γ such that both subgroups G_{Γ_1} and G_{Γ_2} are infinite.*

Definition 2.17. Given a finitely generated group G with the Cayley graph $K(G, S)$ equipped with the path metric d_S and a collection \mathbb{P} of subgroups of G , one can construct the *coned-off Cayley graph* $\hat{K}(G, S, \mathbb{P})$ as follows: For each left coset gP where $P \in \mathbb{P}$, add a vertex v_{gP} , we call *cone-vertex*, to the Cayley graph $K(G, S)$ and for each element x of gP , add an edge $e(x, v_{gP})$, we call *cone-edge*, of length 1 from x to the vertex v_{gP} . This results in a metric space that may not be proper (i.e. closed balls need not be compact). If there exists $\delta > 0$ such that $\hat{K}(G, S, \mathbb{P})$ is δ -hyperbolic, then G is *weakly hyperbolic* relative to the collection \mathbb{P} .

Definition 2.18. Let Γ be a finite simplicial graph with vertex set S and Σ_Γ the associated Davis complex. Then the 1-skeleton $\Sigma_\Gamma^{(1)}$ of Σ_Γ is a Cayley graph of the right-angled Coxeter group G_Γ . We define the associated *star coned-off graph* of G_Γ , denoted $\hat{K}_s(G_\Gamma)$, to be the coned-off graph of $\Sigma_\Gamma^{(1)}$ with respect to the collection of all star subgroups of G_Γ . We denote the path metric on $\hat{K}_s(G_\Gamma)$ by \hat{d}_s .

Similarly, we define the associated *join coned-off graph* of G_Γ , denoted $\hat{K}_j(G_\Gamma)$, to be the coned-off graph of $\Sigma_\Gamma^{(1)}$ with respect to the collection of all join subgroups of G_Γ . We also denote the path metric on $\hat{K}_j(G_\Gamma)$ by \hat{d}_j .

Since a star subgroup is always a join subgroup, we can consider the star coned-off graph as a subgraph of the join coned-off graph.

2.3. Hyperplanes, contact graphs, and connection to star coned-off graphs.

Definition 2.19. Let Σ be a CAT(0) cube complex. We consider the equivalence relation on the set of midcubes of cubes generated by the rule that two midcubes are related if they share a face. A *hyperplane*, H , is the union of the midcubes in a single equivalence class. We define the *support* of a hyperplane H , denoted $N(H)$, to be the union of cubes which contain midcubes of H .

Two hyperplanes H_1 and H_2 *cross* if there is a 2-cube whose two distinct midlines are contained in H_1 and H_2 , respectively. This is denoted by $H_1 \perp H_2$. The hyperplanes H_1 and H_2 *osculate* if they do not cross and there exist distinct 1-cubes c and c' intersecting H_1 and H_2 , respectively, such that c and c' have a common 0-cube. In other words, H_1 and H_2 osculate if $N(H_1) \cap N(H_2) \neq \emptyset$ and H_1 and H_2 do not cross. If H_1 and H_2 either cross or osculate, then they *contact*, denoted by $H_1 \not\perp H_2$. Note that $H_1 \not\perp H_2$ if and only if no hyperplane H separates H_1 from H_2 .

The *contact graph* $\mathcal{C}\Sigma$ of Σ is the graph whose vertices are the hyperplanes of Σ , with hyperplanes H_1 and H_2 joined by an edge if and only if $H_1 \not\perp H_2$. Equivalently, $\mathcal{C}\Sigma$ is the nerve of the covering of Σ by the set of hyperplane supports. We denote the path metric on $\mathcal{C}\Sigma$ by d .

Theorem 2.20 (Hagen [Hag13]). *Let Σ be a CAT(0) cube complex such that every clique in $\mathcal{C}\Sigma$ is of uniformly bounded cardinality. Let g be an element in $\text{Aut}(\Sigma)$. Then one of the following holds:*

- (1) *There exists $n > 0$ and a hyperplane H such that $g^n H = H$.*
- (2) *There exists a subgraph $K_{\infty, \infty} \subset \mathcal{C}\Sigma$ stabilized by g . In this case, g is not a rank-one element.*
- (3) *The element g has a quasigeodesic axis in $\mathcal{C}\Sigma$. (i.e. g acts as a loxodromic isometry of $\mathcal{C}\Sigma$.)*

Remark 2.21. Let Γ be a finite simplicial graph with vertex set S and Σ_Γ the associated Davis complex. Each hyperplane in Σ_Γ separates Σ_Γ into two convex sets. It follows that the distance between a pair of vertices with respect to the metric d_S equals the number of hyperplanes in Σ_Γ separating those vertices.

For a generator v , let e_v denote the edge from the basepoint 1 to the vertex v . Any edge in Σ_Γ determines a unique hyperplane, namely the hyperplane containing the midpoint of that edge. Denote by H_v the hyperplane containing the midpoint of e_v .

For a cube in Σ_Γ , all of the parallel edges are labeled by the same generator v . It follows that all of the edges crossing a hyperplane H have the same label v , and we call this a hyperplane of *type* v . Obviously, if two hyperplanes with the types v_1 and v_2 cross, then v_1 and v_2 commute. Since G_Γ acts transitively

on edges labeled v , a hyperplane is of type v if and only if it is a translate of the standard hyperplane H_v . Obviously, the star subgroup $G_{St(v)}$ is the stabilizer of the hyperplane H_v and $G_{St(v)}$ can also be considered as vertices of the support $N(H_v)$ of H_v . Therefore, each cone-vertex in $\hat{K}_s(G_\Gamma)$ can be related to some suitable hyperplane.

The idea for the following lemma comes from Lemma 3.1 in [BC12]. Moreover, the proof of the following lemma is almost identical to the proof of that lemma. Therefore, we here just copy the proof Lemma 3.1 in [BC12] with slight changes that are suitable to the case of RACGs.

Lemma 2.22. *Let $H_1 = g_1 H_v$ and $H_2 = g_2 H_w$. Then*

- (1) H_1 intersects H_2 if and only if v, w commute and $g_1^{-1} g_2 \in G_{St(v)} G_{St(w)}$.
- (2) There is a hyperplane H_3 intersecting both H_1 and H_2 if and only if there is u in $St(v) \cap St(w)$ such that $g_1^{-1} g_2 \in G_{St(v)} G_{St(u)} G_{St(w)}$.

Proof. Without loss of generality, we may assume that $H_1 = H_v$ and $H_2 = g H_w$.

- (1) If v, w commute, they span a cube in the Davis complex Σ_Γ , hence H_v and H_w intersect. Suppose $g = ab$, with $a \in G_{St(v)}$, $b \in G_{St(w)}$. Then $H_v = a H_v$ and $H_w = b H_w$, so reflecting by a , we see that H_v intersects $g H_w$. Conversely, suppose H_v intersects $g H_w$ in a cube C . Then C contains edges of type v and of type w hence v and w must commute. Moreover, C is a translate $C = h C_0$ of a cube C_0 at the basepoint 1 containing the edges e_v and e_w . Since e_v and $h e_v$ both intersect H_v , h lies in $G_{St(v)}$. Since $g e_w$ and $h e_w$ both intersect $g H_w$, $h^{-1} g$ lies in $G_{St(w)}$. Thus, $g \in G_{St(v)} G_{St(w)}$.
- (2) If u in $St(v) \cap St(w)$ and $g = abc \in G_{St(v)} G_{St(u)} G_{St(w)}$, then H_v and $b H_w = b c H_w$ both intersect $H_u = b H_u$. Reflecting by a , we see that H_v and $g H_w$ both intersect $a H_u$. Conversely, suppose that $H_3 = h H_u$ intersects both H_1 and H_2 . By part (1), u must commute with both v and w , so u lies in $St(v) \cap St(w)$. Also by part (1), $h \in G_{St(v)} G_{St(u)}$ and $h^{-1} g \in G_{St(u)} G_{St(w)}$, so $g \in G_{St(v)} G_{St(u)} G_{St(w)}$. □

Theorem 2.23 (Theorem 4.1 in [Hag14]). *Let Γ be a finite simplicial graph and Σ_Γ the associated Davis complex. The contact graph $\mathcal{C}\Sigma_\Gamma$ of Σ_Γ is quasi-isometric to a tree.*

Theorem 2.24 (Hagen [Hag14]). *Let Γ be a finite simplicial graph, Σ_Γ the associated Davis complex, and $\mathcal{C}\Sigma_\Gamma$ the contact graph of Σ_Γ . Let $p : \mathcal{C}\Sigma_\Gamma \rightarrow \hat{K}_s(G_\Gamma)$ be the map that sends each vertex of $\mathcal{C}\Sigma_\Gamma$ to the cone-vertex over the corresponding hyperplane. Moreover, each edge joining a pair of adjacent vertices in $\mathcal{C}\Sigma_\Gamma$ is mapped linearly onto the length 2-path connecting corresponding cone-vertices in $\hat{K}_s(G_\Gamma)$. Then the map p is a G_Γ -equivariant*

quasi-isometry. In particular, the join coned-off graph $\hat{K}_s(G_\Gamma)$ is quasi-isometric to a tree and G_Γ is weakly hyperbolic relative to the collection of all star subgroups of G_Γ .

3. LOXODROMIC ISOMETRIES AND PURELY LOXODROMIC SUBGROUPS

3.1. Formal definitions. As we discussed before, the star coned-off graph $\hat{K}_s(G_\Gamma)$ of a right-angled Coxeter group G_Γ can be consider as a subgraph of its join coned-off graph $\hat{K}_j(G_\Gamma)$. We now prove that the inclusion $i : \hat{K}_s(G_\Gamma) \hookrightarrow \hat{K}_j(G_\Gamma)$ is a G_Γ -equivariant quasi-isometry.

Lemma 3.1. *In the star coned-off graph $\hat{K}_s(G_\Gamma)$, the distance between any two points in a left coset of a join subgroup is at most 4 with respect to the metric \hat{d}_s .*

Proof. Let $\Gamma_1 * \Gamma_2$ be an arbitrary induced join subgraph of Γ and $g_0 G_{\Gamma_1 * \Gamma_2}$ an arbitrary left coset of the join subgroup $G_{\Gamma_1 * \Gamma_2}$. We will prove that the distance between any two points in $g_0 G_{\Gamma_1 * \Gamma_2}$ is at most 4 with respect to the metric \hat{d}_s .

Choose an arbitrary vertex s of Γ_2 . Then Γ_1 is a subgraph of the star graph $St(s)$. Therefore, G_{Γ_1} is a subgroup of the star subgroup $G_{St(s)}$. This implies that each point in a left coset $g G_{\Gamma_1}$ is connected to the cone vertex $v_{g G_{St(s)}}$ by a cone edge $\hat{K}_s(G_\Gamma)$. Thus, the distance between any two points in a left coset $g G_{\Gamma_1}$ is at most 2 with respect to the metric \hat{d}_s . Similarly, the distance between any two points in a left coset $g G_{\Gamma_2}$ is also bounded by 2 with respect to the metric \hat{d}_s . We observe that

$$g_0 G_{\Gamma_1 * \Gamma_2} = g_0 (G_{\Gamma_1} \times G_{\Gamma_2}) = g_0 \bigcup_{g \in G_{\Gamma_2}} g G_{\Gamma_1} = \bigcup_{g \in G_{\Gamma_2}} g_0 g G_{\Gamma_1} = \bigcup_{h \in g_0 G_{\Gamma_2}} h G_{\Gamma_1}.$$

Since the \hat{d}_s -distance between any two points in left coset $h G_{\Gamma_1}$ is at most 2 and the \hat{d}_s -distance $\hat{d}_s(h, h')$ for each h, h' in $g_0 G_{\Gamma_2}$ is also bounded by 2, the \hat{d}_s -distance between any two points in $g_0 G_{\Gamma_1 * \Gamma_2}$ is at most 4 by the triangle inequality. \square

Proposition 3.2. *The inclusion $i : \hat{K}_s(G_\Gamma) \hookrightarrow \hat{K}_j(G_\Gamma)$ is a G_Γ -equivariant quasi-isometry.*

Proof. This is obvious that the inclusion i is a G_Γ -equivariant map. Therefore, we only need to prove that i is a quasi-isometry. We can consider that there are two metrics on the star coned-off graph $\hat{K}_s(G_\Gamma)$: one is itself metric \hat{d}_s and another is induced by the metric \hat{d}_j from the join coned-off graph $\hat{K}_j(G_\Gamma)$. We first prove that for any vertices x and y in $\hat{K}_s(G_\Gamma)$

$$\hat{d}_j(x, y) \leq \hat{d}_s(x, y) \leq 4\hat{d}_j(x, y).$$

Since $\hat{K}_s(G_\Gamma)$ is a subgraph of $\hat{K}_j(G_\Gamma)$, the first part of the above inequality is obvious. Let γ be the shortest path in $\hat{K}_j(G_\Gamma)$ connecting x and y .

Then the length of γ is the distance $\hat{d}_j(x, y)$. Let $x = v_0, v_1, v_2, \dots, v_{n-1}, v_n = y$ be the set of all G_Γ -vertices of γ such that two vertices v_i and v_{i+1} are adjacent or they are both adjacent to a cone vertex of $\hat{K}_j(G_\Gamma)$ in γ . This is obvious that the length of γ (the distance $\hat{d}_j(x, y)$) is bounded below by n . We observe that the distance between v_i and v_{i+1} is exactly 1 with respect to the metric \hat{d}_s if v_i and v_{i+1} are two adjacent vertices in γ . Otherwise, v_i and v_{i+1} both lie in the same left coset of a join subgroup and the distance $\hat{d}_s(v_i, v_{i+1})$ is bounded above by 4 by Lemma 3.1. Therefore, the distance between x and y with respect to the metric \hat{d}_s is at most $4n$ by the triangle inequality. Therefore, $\hat{d}_s(x, y) \leq 4\hat{d}_j(x, y)$. This implies that the inclusion i is a quasi-isometric embedding. Also, each cone vertex in $\hat{K}_j(G_\Gamma)$ is adjacent to a G_Γ -vertex which is also a vertex in $\hat{K}_s(G_\Gamma)$. Therefore, the inclusion i is a quasi-isometry. \square

The following two corollaries are direct results of Proposition 3.2 and Theorem 2.24.

Corollary 3.3. *Let Γ be a finite simplicial graph. The right-angled Coxeter group G_Γ is weakly hyperbolic relative to the collection of all join subgroups of G_Γ .*

Corollary 3.4. *Let Γ be a finite simplicial graph and H a finitely generated subgroup of the right-angled Coxeter group G_Γ . Then the following are equivalent.*

- (1) *Some (any) orbit map from H into the contact graph $\mathcal{C}\Sigma_\Gamma$ is a quasi-isometric embedding.*
- (2) *Some (any) orbit map from H into the star coned-off graph $\hat{K}_s(G_\Gamma)$ is a quasi-isometric embedding.*
- (3) *Some (any) orbit map from H into the join coned-off graph $\hat{K}_j(G_\Gamma)$ is a quasi-isometric embedding.*

In particular, if an element g in the right-angled Coxeter group G_Γ acts as a loxodromic isometry of one of the graphs $\mathcal{C}\Sigma_\Gamma$, $\hat{K}_s(G_\Gamma)$, or $\hat{K}_j(G_\Gamma)$, then g acts as a loxodromic isometry of the remaining two graphs.

Corollary 3.5. *Let Γ be a simplicial connected graph which does not decompose as a nontrivial join and H a finitely generated infinite subgroup of the right-angled Coxeter group G_Γ . An infinite order element g in G_Γ acts on one of (all) subgraphs $\hat{K}_j(G_\Gamma)$, $\hat{K}_s(G_\Gamma)$, and $\mathcal{C}\Sigma_\Gamma$ as a loxodromic isometry iff g is not conjugate into a join group.*

Proof. Assume that g is not conjugate into a join subgroup. Therefore, g is not conjugate into a star subgroup. By Lemma 2.13, $csupp(g^n) = csupp(g)$ for all non-zero integer n . This implies that g^n is not conjugate into a join group for each non-zero integer n . Therefore, no non-zero power of g fixes a hyperplane. Also, g acts on Σ_Γ as a rank-one isometry by Theorem 2.16. This implies that g acts on the contact graph $\mathcal{C}\Sigma_\Gamma$ as a loxodromic isometry

by Theorem 2.20. Therefore, g also acts on join coned-off graph $\hat{K}_j(G_\Gamma)$ and star coned-off graph $\hat{K}_s(G_\Gamma)$ as a loxodromic isometry by Corollary 3.4.

If g is conjugate into a join subgroup, then g fixes some cone vertex in the join coned-off graph $\hat{K}_j(G_\Gamma)$. Therefore, g does not act on the join coned-off graph $\hat{K}_j(G_\Gamma)$ as a loxodromic isometry. By Corollary 3.4, the actions of g on the contact graph $\mathcal{C}\Sigma_\Gamma$ and the star coned-off graph $\hat{K}_s(G_\Gamma)$ are not loxodromic. \square

With the motivation from the Corollaries 3.4 and 3.5 we define loxodromic elements in right-angled Coxeter groups and their purely loxodromic subgroups.

Definition 3.6. Let Γ be a simplicial connected graph which does not decompose as a nontrivial join. An infinite order element g in the right-angled Coxeter group G_Γ is *loxodromic* if g satisfies one of following equivalent conditions:

- (1) g is not conjugate into a join group.
- (2) g acts as a loxodromic isometry on the contact graph $\mathcal{C}\Sigma_\Gamma$.
- (3) g acts as a loxodromic isometry on the join coned-off graph $\hat{K}_j(G_\Gamma)$.
- (4) g acts as a loxodromic isometry on the star coned-off graph $\hat{K}_s(G_\Gamma)$.

An infinite subgroup H of G_Γ is *purely loxodromic* if each infinite order element in H is loxodromic.

By Theorem 2.16, we observe that all loxodromic elements in a right-angled Coxeter group G_Γ act on the Davis complex Σ_Γ as rank-one isometries but the converse is not true. For example, we can choose an infinite order element g of G_Γ such that g lies in some star subgroup but it is not conjugate into a join subgroup $G_{\Gamma_1 * \Gamma_2} = G_{\Gamma_1} \times G_{\Gamma_2}$, where both subgroups G_{Γ_1} and G_{Γ_2} are infinite. Then g still acts on the Davis complex Σ_Γ as a rank-one isometry (see Theorem 2.16) but g is not a loxodromic element.

3.2. Connection to special subgroups, star-free subgroups, and stable subgroups. With a milder condition on subgroups of G_Γ , we have concept of *star-free* subgroups. More precisely, an infinite subgroup H of G_Γ is *star-free* if none of its infinite order elements are conjugate into a star subgroup. We remark again that a purely loxodromic subgroup of G_Γ is star-free, but the converse is false. For example, we can chose Γ as a square labeled cyclically by the vertices a, b, c, d . Then

$$G_\Gamma = \langle a, c \rangle \times \langle b, d \rangle \cong D_\infty \times D_\infty.$$

Note that G_Γ has no loxodromic elements in this case, but any cyclic group generated by cyclically reduced word with full support is star-free. In particular, the cyclic subgroup $\langle abcd \rangle$ is a star-free subgroup.

We now show the connection among parabolic subgroups, star-free subgroups, and purely loxodromic subgroups.

Proposition 3.7. *Let Γ be a simplicial connected graph which does not decompose as a nontrivial join. Let H be a conjugate of a special subgroup induced by a subset S_1 of vertex set of Γ . Then the following are equivalent:*

- (1) *S_1 contains at least two non-adjacent vertices and the distance in Γ between any two elements of S_1 is different from 2.*
- (2) *H is purely loxodromic.*
- (3) *H is star-free.*

Subgroup H satisfying some (all) above condition is a virtually free subgroup.

Proof. Since any purely loxodromic subgroup is star free, then we only need to prove (1) implies (2), and (3) implies (1). Without the loss of generality we can assume that H is a special subgroup. We first prove that (3) implies (1). In fact, if vertices in S_1 are pairwise adjacent, then H is a finite subgroup and then H is not star-free. If H has two vertices u and v with distance 2 in Γ , then $h = uv$ is an infinite order of H which belongs to some star subgroup. Therefore, H is not a star-free subgroups in this case.

We now prove that (1) implies (2). Assume that H is not a purely loxodromic. Then there is an infinite order element h in H that is conjugate to a join subgroup. Then $csupp(h)$ is a subset of the vertex set of some induced join subgraph Γ_1 . Since h is an infinite order element of the special group generated by S_1 , $csupp(h)$ is a subset of S_1 and there are two vertices v_1 and v_2 in $csupp(h)$ that are not adjacent in Γ . Since two non-adjacent vertices v_1 and v_2 both lie in the join subgraph Γ_1 , the distance in Γ between v_1 and v_2 is exactly 2. This is a contradiction. Therefore, H is a purely loxodromic subgroup.

We observe that if S_1 contains at least two non-adjacent vertices and the distance in Γ between any two elements of S_1 is different from 2, then the subgraph induced by S_1 is disconnected and each component is a single point or a clique. Therefore, H is a free product of more than one finite subgroups. This implies that H is a virtually free subgroup. \square

By the above proposition, parabolic purely loxodromic subgroups are always virtually free subgroups. We remark that any infinite subgroup of a purely loxodromic subgroup is also purely loxodromic. Therefore, we conclude that any infinite subgroup which is conjugate into a purely loxodromic special subgroup is also virtually free purely loxodromic subgroup. In general, we will show that a purely loxodromic subgroup is not necessarily conjugate into a purely loxodromic special subgroup. However, in the rest of the paper we will prove that a purely loxodromic subgroups is always virtually free even when it is not conjugate into a purely loxodromic special subgroup.

We now come up with an example of purely loxodromic subgroup which is not conjugate into a purely loxodromic special subgroup. Let Γ be a graph in Figure 1. Then we observe that the distance between any two non-adjacent vertices in Γ is exactly two. Therefore, the group G_Γ does not contains any purely loxodromic parabolic subgroups by Proposition 3.7. Let

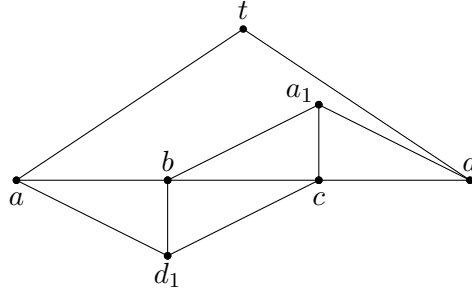


FIGURE 1

$x = (aa_1)(dd_1)(aa_1)$, $y = (dd_1)(aa_1)(dd_1)$, and H a subgroup generated by x and y . Then H is a free subgroup of rank two and H is also a purely loxodromic subgroup (see the following proposition).

Proposition 3.8. *Let Γ be a graph in Figure 1 and H a subgroup generated by $x = (aa_1)(dd_1)(aa_1)$ and $y = (dd_1)(aa_1)(dd_1)$. Then H is a free subgroup of rank two and H is also a purely loxodromic subgroup.*

Proof. Let S be the vertex set of Γ and $T = \{x, y, x^{-1}, y^{-1}\}$. Let $w = u_1 u_2 \cdots u_n$ be an arbitrary freely reduced word in T and \bar{w} be the word obtained from w by replacing x, x^{-1}, y, y^{-1} by their corresponding subwords in G_Γ . We remark that w and \bar{w} both represent the same element in H . We will prove that \bar{w} is a reduced word in G_Γ .

Since w is a freely reduced word in T , then subword of two consecutive elements $u_i u_{i+1}$ in w must lie in

$$\{xx, x^{-1}x^{-1}, yy, y^{-1}y^{-1}, xy, y^{-1}x^{-1}, yx, x^{-1}y^{-1}, x^{-1}y, y^{-1}x, xy^{-1}, yx^{-1}\}.$$

By using the Deletion Condition, we can check that any subwords in \bar{w} that replaces two consecutive elements $u_i u_{i+1}$ in w is reduced. Assume for the contradiction that \bar{w} is not a reduced word in G_Γ . Then using the Deletion Condition, there exists $1 \leq \ell < k \leq 6n$ such that the ℓ^{th} element v_ℓ and the k^{th} element v_k in \bar{w} are labelled by the same generator in S and v_ℓ commutes with all elements between v_ℓ and v_k . We can assume further that no element of \bar{w} between v_ℓ and v_k is labelled by the same generator as v_ℓ and v_k . Also, any subword of \bar{w} that replaces x, y, x^{-1}, y^{-1} has the same support as \bar{w} . Therefore, v_ℓ and v_k must lie in the subword that replaces two consecutive elements $u_i u_{i+1}$ in w . This implies that the subword that replaces two consecutive elements $u_i u_{i+1}$ in w is not reduced. This is a contradiction. Therefore, \bar{w} is a reduced word in G_Γ . This implies that H is a free subgroup of rank 2 and $|h|_S = 6|h|_T$ for each element h in H . This fact also implies that if h is cyclically reduced in (H, T) , then h is also cyclically reduced in (G_Γ, S) .

We now assume for the contradiction that H is not a purely loxodromic subgroup. Then there is a nontrivial element h that is conjugate into a join

subgroup. We can assume that h is cyclically reduced in (H, T) . Therefore, h is also cyclically reduced in (G_Γ, S) and h lies in a join subgroup. Therefore, the support $\text{supp}(h) = \{a, a_1, d, d_1\}$ must lie in the vertex set of some nontrivial join subgraph $\Gamma' = \Gamma_1 * \Gamma_2$. Since the subgraph of Γ induced by $\text{supp}(h)$ is not a join, then $\text{supp}(h) = \{a, a_1, d, d_1\}$ must lie entirely in Γ_1 or Γ_2 (say Γ_1). Therefore, $\text{supp}(h) = \{a, a_1, d, d_1\}$ lies entirely in the star of some vertex in Γ_2 . We can check easily that this is a contradiction. Therefore, H is a purely loxodromic subgroup. \square

In the rest of the paper, we will prove that all purely loxodromic subgroups in RACGs are stable. However, the converse is not true. For example, a cyclic subgroup H of a right-angled Coxeter group G_Γ generated by a rank-one isometry g is stable but H is not a purely loxodromic subgroup when g is conjugate into a star subgroup. We can also construct a non-virtually cyclic stable subgroup which is not purely loxodromic as follows. Let Γ be a finite, simplicial, connected graph which has no separating vertices or edges and no embedded cycles of length four. We also assume that Γ contains an embedded cycle C of length more than four. Then the right-angled Coxeter group G_Γ is a one-ended hyperbolic group (see Theorem 8.7.2 and Corollary 12.6.3 in [Dav08]) and the special subgroup G_C is a non-virtually cyclic quasiconvex subgroup of G_Γ . Therefore, G_C is a non-virtually cyclic stable subgroup. It is obvious that the vertex set of C does not satisfy conditions in Proposition 3.7. Therefore, G_C is not a purely loxodromic subgroup.

4. DUAL VAN KAMPEN DIAGRAMS FOR RIGHT-ANGLED COXETER GROUPS

In this section, we construct dual van Kampen diagrams for right-angled Coxeter groups which are almost identical to dual van Kampen diagrams for right-angled Artin groups constructed in [Kim08]. In [KMT], Koberda-Mangahas-Taylor used dual van Kampen diagrams for right-angled Artin groups to study the geometry of their purely loxodromic subgroups and star-free subgroups. In Sections 4 and 5 of this article, we will follow the same strategy as in [KMT] to study the geometry of purely loxodromic subgroups and star-free subgroups in right-angled Coxeter groups.

4.1. Formal definition. We now develop dual van Kampen diagrams for right-angled Coxeter groups. The key ingredient for constructing such diagrams for RACGs which are similar to ones for RAAGs is the similarity between Davis complexes and universal covers of Salvetti complexes.

Let Γ be a finite simplicial graph with the vertex set S . Let w be a word representing the trivial element in G_Γ . A *dual van Kampen diagram* Δ for w in G_Γ is an oriented disk D together with a collection \mathcal{A} , properly embedded arcs in general position, satisfying the following:

- (1) Each arc of \mathcal{A} is labeled by an element of S . Moreover, if two arcs of \mathcal{A} intersect then the generators corresponding to their labels are adjacent in Γ .

- (2) With its induced orientation, ∂D represents a cyclic conjugate of the word w in the following manner: there is a point $*$ $\in \partial D$ such that w is spelled by starting at $*$, traversing ∂D according to its orientation, and recording the labels of the arcs of \mathcal{A} it encounters

We think of the boundary of D as subdivided into edges and labeled according to the word w . In this way, each arc of \mathcal{A} corresponds to two letters of w which are represented by edges on the boundary of D . While not required by the definition, it is convenient to restrict our attention to *tight* dual van Kampen diagrams, in which arcs of \mathcal{A} intersect at most once.

In comparison with dual van Kampen diagrams for RAAGs, the only difference from dual van Kampen diagrams for RACGs is we do not need a direction equipped on each embedded arc of \mathcal{A} and each edge of ∂D . The key reason for this difference is each edge of universal covers of Salvetti complexes is equipped with a direction while each edge of Davis complexes is not.

We now show the way to construct the dual van Kampen diagram for an identity word w in a right-angled Coxeter groups. Let $\tilde{\Delta} \subset S^2$ be a (standard) van Kampen diagram for w , with respect to a standard presentation of G_Γ . Consider $\tilde{\Delta}^*$, the dual of $\tilde{\Delta}$ in S^2 , and name the vertex which is dual to the face $S^2 - \tilde{\Delta}$ as v_∞ . Then for a sufficiently small ball $B(v_\infty)$ around v_∞ , $\tilde{\Delta}^* - B(v_\infty)$ can be considered as a dual van Kampen diagram with a suitable choice of the labeling map. Therefore a dual van Kampen diagram exists for any word w representing the trivial element in G_Γ . Conversely, a van Kampen diagram $\tilde{\Delta}$ for a word can be obtained from a dual van Kampen diagram Δ by considering the dual complex again. So, the existence of a dual van Kampen diagram for a word w implies that w represents the trivial element in G_Γ .

4.2. Surgery and subwords. Let Γ be a finite simplicial graph with the vertex set S . Starting with a dual van Kampen diagram Δ with a disk D and collection \mathcal{A} of embedded arcs in D for an identity word w . Suppose that γ is a properly embedded arc in Δ which is either an arc of \mathcal{A} or transverse to the arcs of \mathcal{A} . Traversing γ in some direction and recording the labels of those arcs of \mathcal{A} that cross γ spells a word y in the standard generators. We say the word y is obtained from *label reading* along γ with the chosen direction.

In particular, starting with a subword w' of w , any oriented arc of D which begins at the initial vertex of w' and ends at the terminal vertex of w' produces a word y via label reading such that $w' = y$ in G_Γ . To see this, we observe that the arc γ cuts the disk D into two disks D_1 and D_2 , one of which (say D_1) determines the homotopy (and sequence of moves) to transform the word w' into y . In other word, the disk D_1 along with arcs from \mathcal{A} forms a dual van Kampen diagram for the word $w'\overline{y}$, and we say that this diagram is obtained via *surgery* on Δ . It is straightforward that if the arc γ is labelled by a vertex v in S , then w' represents an element

in the star subgroup $G_{St(v)}$. You can see the following lemma for a precise statement.

Lemma 4.1. *Suppose an arc of \mathcal{A} in a dual van Kampen diagram Δ for the identity word w cuts off the subword w' , i.e., $w \equiv sw'vt$, where s , w' , and t are subwords and v is the letter at the ends of the arc. Then w' represents a group element in the star subgroup $G_{St(v)}$.*

If a subword in a dual van Kampen diagram has the property that no two arcs emanating from it intersect, this subword is *combed* in the dual van Kampen diagram. We remark that this such type of subword was also defined for dual van Kampen diagrams for RAAGs in [KMT] and it played an important role to study some certain types of subgroups of RAAGs. In Sections 4 and 5 of this article, we are following the same strategy in [KMT] to study subgroups of RACGs. Therefore, the property of being combed will be important in these sections.

Lemma 4.2. *Suppose w is a word representing the identity and b is a subword of w , so w is the concatenation of words a , b , and c . Let Δ be a dual van Kampen diagram for w .*

Then there exists a word b' obtained by re-arranging the letters in b , such that $b' = b$ and there exists a dual van Kampen diagram Δ' for $ab'c$ in which b' is combed, arcs emanating from b' have the same endpoint in the boundary subword ca as their counterpart in b , and arcs that both begin and end in ca are unchanged in Δ' .

Furthermore, there exists a word b'' obtained by deleting letters in b' , such that $b'' = b$ and there exists a dual van Kampen diagram Δ'' for $ab''c$ which is precisely Δ' without the arcs corresponding to the deleted letters.

The above lemma is identical to Lemma 3.2 in [KMT] for RAAGs. Moreover, we observe that the proof of Lemma 3.2 in [KMT] can be applied to prove the above lemma. Therefore, the reader can see the proof of Lemma 3.2 in [KMT] to obtain the proof of the above lemma.

4.3. Reducing diagrams. In subsection 3.5 in [KMT], Koberda-Mangahas-Taylor introduce reducing diagrams and some related concepts to study words in RAAGs as well as paths in universal covers of Salvetti complexes. We observed that these concepts are also well-defined for the case of RACGs and they can also help us studying words in RACGs as well as paths in Davis complexes. Therefore, we just copy most of subsection 3.5 in [KMT] and the reader can verify easily that these materials fit well for the case of RACGs.

Let h be a word in the vertex generators of G_Γ , which is not assumed to be reduced in any sense. Let w denote a reduced word in the vertex generators which represents the same group element as h does. Then, the word $h\bar{w}$ represents the identity in G_Γ and so it is the boundary of some dual van Kampen diagram Δ . (Here \bar{w} denotes the inverse of the word w .) In this way, the boundary of Δ consist of two words h and \bar{w} . We sometimes refer to a dual van Kampen diagram constructed in this way as a *reducing*

diagram as it represents a particular way of reducing h to the reduced word w . For such dual van Kampen diagrams, ∂D is divided into two subarcs (each a union of edges) corresponding to the words h and w , we call these subarcs the h and w subarcs, respectively.

Suppose that Δ is a dual van Kampen diagram that reduces h to the reduced word w . Since w is already a reduced word, no arc of \mathcal{A} can have both its endpoints on the w subarc of ∂D . Otherwise, one could surger the diagram to produce a word equivalent to w with fewer letters. Hence, each arc of \mathcal{A} either has both its endpoints on the subarc of ∂D corresponding to h , or it has one endpoint in each subarc of ∂D . In the former case, we call the arc (and the letters of h corresponding to its endpoints) *noncontributing* since these letters do not contribute to the reduced word w . Otherwise, the arc is called *contributing* (as is the letter of h corresponding the endpoint contained in the h subarc of ∂D). If the word h is partitioned into a product of subwords abc , then the *contribution of the subword b to w* is the set of letters in b which are contributing. We remark that whether a letter of h is contributing or not is a property of the fixed dual van Kampen diagram that reduces h to w .

5. THE GEOMETRY OF SUBGROUPS OF RIGHT-ANGLED COXETER GROUPS

5.1. The geometry of star free subgroups. Recall that a nontrivial subgroup H of G_Γ is *star-free* if each infinite order element in H is not conjugate into a star subgroup. We now assume that H is a finitely generated star-free subgroup of G_Γ with a finite generating set T . Therefore, each element $h \in H$ can be expressed as a geodesic word in H , that is, $h = h_1 h_2 \cdots h_n$ such that $h_i \in T$ and n is minimal. We use a dual van Kampen diagram with boundary word $(h_1 h_2 \cdots h_n) h^{-1}$, where h and each h_i are written as reduced words in G_Γ . In other words, we concatenate the reduced word representatives for the h_i to obtain a word representing $h = h_1 \cdots h_n$ and consider a reducing diagram for this word. With our choices fixed, we call such a reducing diagram for h simply a dual van Kampen diagram for $h \in H$.

The following lemma is identical to the Lemma 4.1 in [KMT] for RAAGs. Moreover, their proofs are almost identical except there is some small extra step at the end of the proof of the following lemma.

Lemma 5.1. *Suppose H is a finitely generated, star-free subgroup of G_Γ . There exists $D = D(H)$ with the following property: If in a dual van Kampen diagram for $h \in H$, a letter in h_i is connected to a letter in h_j ($i < j$), then $j - i < D$.*

Proof. Suppose in a dual van Kampen diagram for $h \in H$, a letter g in h_i is connected to another letter g in h_j . By Lemma 4.1, $h_i \cdots h_j = \sigma M \tau$, where M is in the star of g , and σ, τ are a prefix of h_i and suffix of h_j respectively.

Therefore, if the lemma is false, there a sequence of reduced-in- H words

$$h_{i(t)}^{(t)} \cdots h_{j(t)}^{(t)} = \sigma_t M_t \tau_t$$

as above, with $j(t) - i(t)$ strictly increasing. Because Γ is finite and H is finitely generated, we may pass to a subsequence so that the M_t are in the star of the same generator v , and furthermore we have constant $\sigma_t = \sigma$ and $\tau_t = \tau$, while $M_t \neq M_s$ for $s \neq t$. Therefore, for each $t \geq 2$, element

$$k_t = (h_{i(t)}^{(t)} \cdots h_{j(t)}^{(t)}) (h_{i(1)}^{(1)} \cdots h_{j(1)}^{(1)})^{-1} = \sigma M_t M_1^{-1} \sigma^{-1}$$

is nontrivial element in the subgroup $H \cap \sigma G_{St(v)} \sigma^{-1}$. Moreover, $k_t \neq k_s$ for any $2 \leq t < s$.

Assume that k_{t_0} is infinite order for some $t_0 \geq 2$. Then, H is not a star-free subgroup which is a contradiction. We now assume that the order of all k_t are two. Since $k_t \neq k_s$ for any $2 \leq t < s$, we can choose two different elements k_{t_1} and k_{t_2} which do not commute. Therefore, the order of the group element $k_{t_1} k_{t_2}$ is not two. This implies that $k_{t_1} k_{t_2}$ is an infinite order element in the subgroup $H \cap \sigma G_{St(v)} \sigma^{-1}$. Then, H is not a star-free subgroup which is a contradiction. \square

The following lemma is identical to the Lemma 4.2 in [KMT] for RAAGs. Moreover, the proof of the below lemma almost follows the same line argument as in the proof of Lemma 4.2 in [KMT]. Here we only need to replace Lemmas 3.2, 4.1 in the proof of Lemma 4.2 in [KMT] by Lemmas 4.2, 5.1 of this paper respectively to obtain the proof of the following lemma.

Lemma 5.2. *Suppose H is a finitely generated, star-free subgroup of G_Γ and D is a constant as in Lemma 5.1. Let $h_i \cdots h_j$ be a subword of $h = h_1 \cdots h_n$ reduced in H as above. Then the element $h_i \cdots h_j \in G_\Gamma$ may be written as a concatenation of three words $\sigma W \tau$, where the letters occurring in σ are a subset of the letters occurring in $h_{i-D} \cdots h_{i-1}$ when $i > D$, and in $h_1 \cdots h_{i-1}$ otherwise; the letters occurring in τ are a subset of the letters occurring in $h_{j+1} \cdots h_{j+D}$ when $j \leq n - D$ and in $h_{j+1} \cdots h_n$ otherwise; and the letters occurring in W are exactly the letters occurring in $h_i \cdots h_j$ which survive in the word h after it is reduced in G_Γ .*

The following lemma is a key lemma to prove that the star-free subgroup H is undistorted in G_Γ . This lemma is identical to the Lemma 4.3 in [KMT] for RAAGs and its proof again almost follows the same line argument as in the proof of Lemma 4.3 in [KMT]. Here we only need to replace Lemma 4.2 in the proof of Lemma 4.3 in [KMT] by Lemma 5.2 of this paper to obtain the proof of the following lemma.

Lemma 5.3. *Given H a finitely generated, star-free subgroup of G_Γ , there exists $K = K(H)$ such that, if $h_i \cdots h_j$ is a subword of a reduced word for h in H which contributes nothing to the reduced word for h in G_Γ , then $j - i < K$.*

The following proposition is a direct result of Lemma 5.3.

Proposition 5.4. *Finitely generated star-free subgroups are undistorted.*

The proof of the following proposition is almost identical to the proof of Theorem 53 in [KK14]. We recall a proof with a slight modification for the convenience of the reader.

Proposition 5.5. *Star-free subgroups are virtually free.*

Proof. We first assume that H is torsion free. We will prove that H is a free subgroup by induction on the number of vertices of Γ . Since H is a torsion free star-free subgroup, for each vertex v of Γ and g in G_Γ

$$H \cap gG_{St(v)}g^{-1} = \{1\}.$$

For the base case $\Gamma = v$, $G_\Gamma = \mathbb{Z}_2$ and $H = \{1\}$. Therefore, the result in this case is obvious. For the inductive step, choose a vertex v of Γ and let Γ_v be the induced subgraph of Γ generated by all vertices of Γ except v . We observe that $G_\Gamma = G_{St(v)} *_{G_{Lk(v)}} G_{\Gamma_v}$. By standard Bass-Serre Theory, we see that H acts on the corresponding Bass-Serre tree with trivial edge stabilizer. Therefore, there exists a (possibly infinite) collection of subgroups $\{H_i\}$ with each H_i conjugate to G_{Γ_v} in G_Γ such that H is a free product of subgroups H_i with possibly an additional free factor. Since H_i is conjugate into G_{Γ_v} and Γ_v has fewer vertices than Γ , we see that H is free by induction.

We now assume that H is not torsion free. Let G_1 be a finite-index torsion free subgroup of G_Γ and $H_1 = G_1 \cap H$ (see [DL98] for a construction of group G_1). Then, H_1 is a torsion free star-free subgroup of H with a finite index in H . Also, H_1 is a free subgroup by the above argument. This implies that H is a virtually free subgroup. \square

5.2. Geometric embedding properties of purely loxodromic subgroups. Assume the graph Γ is a connected graph which does not decompose as a nontrivial join. A nontrivial subgroup H of G_Γ is *N-join-busting* if, for any reduced word w representing h in H , and any join subword $\beta \leq w$, the length of w is bounded above by N .

By using almost the same line argument as in Section 5 in [KMT], we obtained the Proposition 5.6 as below. We remark that the Proposition 5.6 is identical to the Theorem 5.2 in [KMT]. However, we need to use van Kampen diagrams for RACGs instead of van Kampen for RAAGs in [KMT]. We also use Lemmas 4.2, 5.1, 5.2, and Corollary 3.5 of this paper in stead of Lemmas 3.2, 4.1, 4.2, and Theorem 2.2 in [KMT] respectively.

Proposition 5.6. *Let Γ be a simplicial connected graph which does not decompose as a nontrivial join and H a finitely generated purely loxodromic subgroup of the right-angled Coxeter group G_Γ . There exists an $N = N(H)$ such that H is N -join-busting.*

In the Proposition 5.7 as below, we prove the stability of N -join-busting in RACGs. This proposition is identical to Corollary 6.2 in [KMT]. The

proof of Proposition 5.7 follows almost the same line argument as in Section 6 in [KMT]. However, we need to use van Kampen diagrams for RACGs instead of van Kampen for RAAGs in and we use Proposition 5.4 in this paper in stead of Proposition 4.4 in [KMT].

Proposition 5.7. *Let Γ be a simplicial connected graph which does not decompose as a nontrivial join and H a finitely generated purely loxodromic subgroup of the right-angled Coxeter group G_Γ . If H is N -join-busting for some N , then H is stable in G_Γ .*

Theorem 5.8. *Let Γ be a simplicial connected graph which does not decompose as a nontrivial join and H a finitely generated infinite subgroup of the right-angled Coxeter group G_Γ . Then the following are equivalent:*

- (1) *H is purely loxodromic.*
- (2) *Some (any) orbit map from H into the contact graph \mathcal{CS}_Γ is a quasi-isometric embedding.*
- (3) *Some (any) orbit map from H into the star coned-off graph $\hat{K}_s(G_\Gamma)$ is a quasi-isometric embedding.*
- (4) *Some (any) orbit map from H into the join coned-off graph $\hat{K}_j(G_\Gamma)$ is a quasi-isometric embedding.*

Proof. Let S be the vertex set of Γ and d_S the word metric on the Cayley graph $\Sigma_\Gamma^{(1)}$. By Corollary 3.4, it is sufficient to prove the statements (1) and (4) are equivalent. Assume that H is purely loxodromic subgroup. Then there exists an $N = N(H)$ such that H is N -join-busting by Proposition 5.6. Moreover, H is also a star-free subgroup so H is a undistorted subgroup of G_Γ by Proposition 5.4. Therefore, in order to prove that some (any) orbit map from H into the join coned-off graph $\hat{K}_j(G_\Gamma)$ is a quasi-isometric embedding, it is sufficient to prove that for each h in H

$$\hat{d}_j(e, h) \leq d_S(e, h) \leq N\hat{d}_j(e, h).$$

Since the Cayley graph $\Sigma_\Gamma^{(1)}$ is a subgraph of the join coned-off graph $\hat{K}_j(G_\Gamma)$, then the first part of the above inequality is obvious. Also there is a positive integer $k \leq \hat{d}_j(e, h)$ such that h can be written as $g_1 g_2 \cdots g_k$, where each g_i belong to some join subgroup. Therefore, $d_S(e, h) \leq Nk$ since H is N -join-busting. This implies that $d_S(e, h) \leq N\hat{d}_j(e, h)$. Thus, some (any) orbit map from H into the join coned-off graph $\hat{K}_j(G_\Gamma)$ is a quasi-isometric embedding.

Assume that H is not a purely loxodromic subgroup. Then there is an infinite order element h in H that is conjugate into some join subgroup. Therefore, h fixes some coned vertex v in $\hat{K}_j(G_\Gamma)$. This implies that the orbit Hv is not a quasi-isometric embedding in $\hat{K}_j(G_\Gamma)$. Therefore, the statements (1) and (4) are equivalent. \square

6. SUBGROUP DIVERGENCE OF PURELY LOXODROMIC SUBGROUPS IN RIGHT-ANGLED COXETER GROUPS

6.1. Lower bound on the subgroup divergence. The following proposition is a direct result of Propositions 5.6 and 5.7.

Proposition 6.1. *Let Γ be a simplicial, finite, connected graph with the vertex set S such that Γ does not decompose as a nontrivial join. Let H be a finitely generated purely loxodromic subgroup. Then there is a positive constant M such that every geodesic in the Cayley graph $\Sigma_\Gamma^{(1)}$ connecting points in H lies in the M -neighborhood of H .*

Theorem 6.2. *Let Γ be a simplicial, finite, connected graph with the vertex set S such that Γ does not decompose as a nontrivial join. Let H be a finitely generated purely loxodromic subgroup of G_Γ . Then the subgroup divergence of H in G_Γ is at least quadratic.*

Proof. By Proposition 6.1, there is a positive integer M such that every geodesic in the Cayley graph $\Sigma_\Gamma^{(1)}$ connecting points in H lies in M -neighborhood of H . By Proposition 5.6, there is another positive integer N such that for any reduced word w representing $h \in H$, and any join subword w' of w , we have $\ell(w') \leq N$. Let $\{\sigma_\rho^n\}$ be the subspace divergence of H in the Cayley graph $\Sigma_\Gamma^{(1)}$. We will prove that for each $n \geq 9$ and $\rho \in (0, 1]$

$$\sigma_\rho^n(r) \geq \left(\frac{r-1}{3N+1}\right)(\rho r - 3N) - 2r \text{ for each } r > \frac{2M+3N+2}{\rho}.$$

If there is no pair of $u, v \in \partial N_r(H)$ such that $d_r(u, v) < \infty$ and $d(u, v) \geq nr$, then $\sigma_\rho^n(r) = \infty$ and we are done. Otherwise, let u and v be an arbitrary pair of points in $\partial N_r(H)$ such that $d_r(u, v) < \infty$ and $d_S(u, v) \geq nr$. Let γ be an arbitrary path that lies outside the ρr -neighborhood of H connecting u and v . We will prove that the length of γ is bounded below by $\left(\frac{r-1}{3N+1}\right)(\rho r - 3N) - 2r$.

Let γ_1 be a geodesic of length r in $\Sigma_\Gamma^{(1)}$ connecting u and some point x in H . Let γ_2 be another geodesic of length r in $\Sigma_\Gamma^{(1)}$ connecting v and some point y in H . Let α be a geodesic in $\Sigma_\Gamma^{(1)}$ connecting x and y . Then α lies in the M -neighborhood of H . Choose a positive integer m such that $r < (3N+1)m+1 < 2r$.

Since $d_S(x, y) \geq d_S(u, v) - 2r \geq (n-2)r \geq 7r$, there is a subpath α_1 with length $(3N+1)m+1$ of α such that $\alpha_1 \cap (B(x, 2r) \cup B(y, 2r)) = \emptyset$. Also, the lengths of γ_1 and γ_2 are both r . This implies that $(\gamma_1 \cup \gamma_2) \cap N_r(\alpha_1) = \emptyset$. Since $\gamma \cap N_{\rho r}(H) = \emptyset$ and $\alpha_1 \subset N_M(H)$, then $\gamma \cap N_{\rho r-M}(\alpha_1) = \emptyset$. Also, $\rho r - M > \rho r/2$. Thus, $\gamma \cap N_{\rho r/2}(\alpha_1) = \emptyset$. Let $\bar{\gamma} = \gamma_1 \cup \gamma \cup \gamma_2$. Then, $\bar{\gamma} \cap N_{\rho r/2}(\alpha_1) = \emptyset$.

We assume that $\alpha_1 = e_0 w_1 e_1 w_2 \cdots e_m w_m$, where each e_i is an edge labelled by some generator a_i in S , each w_i is a subpath of α_1 of length

exactly $3N$. For each $i \in \{0, 1, 2, \dots, m\}$ let H_i be the hyperplane intersecting e_i and v_i a point in $H_i \cap \bar{\gamma}$. For each $i \in \{1, 2, \dots, m\}$ let β_i be the subpath of $\bar{\gamma}$ connecting v_{i-1} and v_i .

If there is some hyperplane of type b that intersects two hyperplanes H_{i-1} and H_i for some $i \in \{1, 2, \dots, m\}$, then $e_{i-1}w_i$ corresponds to a word w that represents an element in $G_{St(a_{i-1})}G_{St(b)}G_{St(a_i)}$ by Lemma 2.22. Also, the length of w is $3N + 1$. Then w has some reduced spelling that is a product of 3 join words and so one of the join words in this latter spelling has length greater than N . This contradicts the choice of N . Therefore, no hyperplane intersects both H_{i-1} and H_i for each $i \in \{1, 2, \dots, m\}$. Also, $\bar{\gamma}$ lies outside the $(\rho r/2)$ -neighborhood of α_1 . We have the following basic fact:

$$\ell(\beta_i) \geq \rho r/2 + \rho r/2 - \ell(w_i) \geq \rho r - 3N.$$

Therefore,

$$\ell(\gamma) = \ell(\bar{\gamma}) - 2r \geq \sum_{i=1}^m \ell(\beta_i) - 2r \geq m(\rho r - 3N) - 2r \geq \left(\frac{r-1}{3N+1}\right)(\rho r - 3N) - 2r.$$

Thus,

$$\sigma_\rho^n(r) \geq \left(\frac{r-1}{3N+1}\right)(\rho r - 3N) - 2r \text{ for each } r > \frac{2M + 3N + 2}{\rho}.$$

or the subgroup divergence of H in G is at least quadratic. \square

Definition 6.3. Given a graph Γ , define the associated *four-cycle* graph Γ^4 as follows. The vertices of Γ^4 are the induced loops of length four (i.e. four-cycles) in Γ . Two vertices of Γ^4 are connected by an edge if the corresponding four-cycles in Γ share a pair of non-adjacent edges. Given a subgraph K of Γ^4 , we define the *support* of K to be the collection of vertices of Γ (i.e. generators of G_Γ) that appear in the four-cycles in Γ corresponding to the vertices of K . A graph Γ is said to be *CFS* if there exists a component of Γ^4 whose support is the entire vertex set of Γ .

Lemma 6.4. *Let Γ be a simplicial, finite, connected graph with the vertex set S such that Γ does not decompose as a nontrivial join. Let H be a finitely generated purely loxodromic subgroup of G_Γ . There is a positive number K such that the following holds. Let g be an element in G_Γ and $(s_1, t_1), (s_2, t_2)$ two pairs of non-adjacent vertices in a four-cycle of Γ . Let $u_1 = s_1 t_1$ and $u_2 = s_2 t_2$. Then*

$$d_S(gu_1^i u_2^j, H) \geq \frac{|i| + |j|}{K} - |g|_S - 1.$$

Proof. By Proposition 5.6, there is a positive integer N such that for any reduced word w representing $h \in H$, and any join subword w' of w , we have $\ell(w') \leq N$. Let $K = (N + 1)/2$ and we will prove that

$$d_S(gu_1^i u_2^j, H) \geq \frac{|i| + |j|}{K} - |g|_S - 1.$$

Let $m = d_S(gu_1^i u_2^j, H)$. Then there is an element g_1 in G_Γ with $|g_1|_S = m$ and h in H such that $h = gu_1^i u_2^j g_1$. Since $u_1^i u_2^j$ is an element in a join subgroup of G_Γ and $|g_1|_S = m$, then h can be represented by a reduced word w that is a product of at most $(|g|_S + 1 + m)$ join subwords. Also, the length of each join subword of w is bounded above by N . Therefore, the length of w is bounded above by $N(|g|_S + m + 1)$. Also,

$$\ell(w) \geq |u_1^i u_2^j|_S - |g_1|_S - |g|_S \geq 2(|i| + |j|) - m - |g|_S.$$

This implies that

$$2(|i| + |j|) - m - |g|_S \leq N(|g|_S + m + 1).$$

Therefore,

$$d_S(gu_1^i u_2^j, H) = m \geq \frac{2(|i| + |j|)}{N + 1} - |g|_S - \frac{N}{N + 1} \geq \frac{|i| + |j|}{K} - |g|_S - 1.$$

□

Lemma 6.5. *Let Γ be a simplicial, finite, CFS, connected graph with the vertex set S such that Γ does not decompose as a nontrivial join. Let C be a component of Γ^4 whose support is the entire vertex set of Γ . Let H be a finitely generated purely loxodromic subgroup of G_Γ and h an arbitrary element in H . There is a number $L = L(H, C) \geq 1$ such that the following holds. Let $m \geq L^2$ an integer and $u = st$, where (s, t) is a pair of non-adjacent vertices in some four-cycle Q_0 of Γ that corresponds to a vertex in C . There is a path α outside the $(m/L - L)$ -neighborhood of H connecting u^m and hu^m with the length bounded above by Lm .*

Proof. Let $M = \text{diam}(C)$, K the positive integer as in Lemma 6.4 and $k = |h|_S$. Let $L = 2(k + 1)(M + 2) + K + k + M + 1$. Choose a reduced word

$$w = s_1 s_2 \cdots s_k, \text{ where } s_i \in S,$$

that represents element h . Since the support of the component C of Γ^4 is the collection of vertices of Γ , for each $i \in \{1, 2, \dots, k\}$ there is a four-cycle Q_i that corresponds to a vertex of the component C of Γ^4 such that Q_i contains the vertex s_i . Let (a_i, b_i) be a pair of non-adjacent vertices in Q_i , $u_i = a_i b_i$ and $w_i = s_1 s_2 \cdots s_i$. Then the length of each word w_i is bounded above by k , $w_{i+1} = w_i s_{i+1}$, and $w_k = w$ that represents element h .

We now construct a path α_0 outside the $(m/L - L)$ -neighborhood of H connecting u^m and $w_1 u_1^m$ with the length bounded above by $2(M + 2)m$. Since $M = \text{diam } C$, we can choose positive integer $n \leq M$ and $n + 1$ four-cycles P_0, P_1, \dots, P_n that corresponds to a vertex of the component C of Γ^4 such that the following conditions hold:

- (1) $P_0 = Q_0$ contains the pair of non-adjacent vertices (s, t) and let $v_0 = u$.
- (2) $P_n = Q_1$ contains the pair of non-adjacent vertices (a_1, b_1) and let $v_{n+1} = u_1$.

- (3) P_{j-1} and P_j share an pair of non-adjacent vertices (c_j, d_j) , where $j \in \{1, 2, \dots, n\}$ and let $v_j = c_j d_j$.

For each $j \in \{0, 1, 2, \dots, n\}$ let β_j be a path connecting v_j^m and v_{j+1}^m of length $2m$ with vertices

$$v_j^m, v_j^m v_{j+1}^m, v_j^m v_{j+1}^{2m}, \dots, v_j^m v_{j+1}^m, v_j^{m-1} v_{j+1}^m, v_j^{m-2} v_{j+1}^m, \dots, v_{j+1}^m.$$

By Lemma 6.4 the above vertices must lie outside the $(m/K-1)$ -neighborhood of H . Therefore, these vertices also lies outside the $(m/L-L)$ -neighborhood of H . Therefore, β_j is a path outside the $(m/L-L)$ -neighborhood of H connecting v_j^m and v_{j+1}^m . Since $w_1 u_1^m = s_1 u_1^m = u_1^m s_1$, then we can connect u_1^m and $w_1 u_1^m$ by an edge β_{n+1} labelled by s_1 . Let $\alpha_0 = \beta_0 \cup \beta_1 \cup \dots \cup \beta_n \cup \beta_{n+1}$. Then, it is obvious that the path α_0 outside the $(m/L-L)$ -neighborhood of H connecting u^m and $w_1 u_1^m$ with the length bounded above by $2(M+2)m$.

By similar constructions as above, for each $i \in \{1, 2, \dots, k-1\}$ there is a path α_i outside the $(m/L-L)$ -neighborhood of H connecting $w_i u_i^m$ and $w_{i+1} u_{i+1}^m$ with the length bounded above by $2(M+2)m$. We can also construct a path α_k outside the $(m/L-L)$ -neighborhood of H connecting hu_k^m and hu^m with the length bounded above by $2(M+1)m$. Let $\alpha = \alpha_0 \cup \alpha_1 \cup \dots \cup \alpha_k$. Then, it is obvious that the path α outside the $(m/L-L)$ -neighborhood of H connecting u^m and hu^m with the length bounded above by $2(k+1)(M+2)m$. By the choice of L we observe that the length of α is also bounded above by Lm . \square

Theorem 6.6. *Let Γ be a simplicial, finite, connected, CFS graph with the vertex set S such that Γ does not decompose as a nontrivial join. Let H be a finitely generated purely loxodromic subgroup of G_Γ . Then the subgroup divergence of H in G_Γ is exactly quadratic.*

Proof. By Theorem 6.2, the subgroup divergence of H in G_Γ is at least quadratic. Therefore, we only need to prove that the subgroup divergence of H in G_Γ is at most quadratic. Since Γ is a CFS graph, there is a component C of Γ^4 whose support is the entire vertex set of Γ . Let $L = L(H, C)$ be a constant as in Lemma 6.5 and h an arbitrary infinite order group element in H . Since each cyclic subgroup in a CAT(0) group is undistorted, there is a positive integer L_1 such that

$$|h^k|_S \geq \frac{|k|}{L_1} - L_1 \text{ for each integer } k.$$

Let $\{\sigma_\rho^n\}$ be the subspace divergence of H in the Cayley graph $\Sigma_\Gamma^{(1)}$. We will prove that function $\sigma_\rho^n(r)$ is bounded above by some quadratic function for each $n \geq 2$ and $\rho \in (0, 1]$.

Choose a positive integer $m \in [L(L+r), 2L(L+r)]$ and a group element $u = st$, where (s, t) is a pair of non-adjacent vertices in a four-cycle Q_0 of Γ that corresponds to a vertex in C . Then, there is a path α_0 outside the $(m/L-L)$ -neighborhood of H connecting u^m and hu^m with the length bounded above by Lm . It is obvious that the path α_0 also lies outside the

r -neighborhood of H by the choice of m . Choose a positive integer k which lies between $L_1(nr + 16L(L + r) + L_1)$ and $L_1(nr + 16L(L + r) + L_1 + 1)$. Let $\alpha = \alpha_0 \cup h\alpha_0 \cup h^2\alpha_0 \cup \dots \cup h^{k-1}\alpha_0$. Then, α is a path outside the r -neighborhood of H connecting $u^m, h^k u^m$ with the length bounded above by kLm . By the choice of k and m , the length of α is bounded above by $2L_1L^2(L + r)(nr + 16L(L + r) + L_1 + 1)$.

Since $r \leq d_S(u^m, H) \leq 2m$, then there is a path γ_1 outside $N_r(H)$ connecting u^m and some point $x \in \partial N_r(H)$ such that the length of γ_1 is bounded above by $2m$. By the choice of m , the length of γ_1 is also bounded above by $4L(L + r)$. Similarly, there is a path γ_2 outside $N_r(H)$ connecting $h^k u^m$ and some point $y \in \partial N_r(H)$ such that the length of γ_2 is bounded above by $4L(L + r)$. Let $\bar{\alpha} = \gamma_1 \cup \alpha \cup \gamma_2$ then $\bar{\alpha}$ is a path outside $N_r(H)$ connecting x, y and the length of $\bar{\alpha}$ is bounded above by $2L_1L^2(L + r)(nr + 16L(L + r) + L_1 + 1) + 8L(L + r)$. Therefore, for each $\rho \in (0, 1]$

$$d_{\rho r}(x, y) \leq 2L_1L^2(L + r)(nr + 16L(L + r) + L_1 + 1) + 8L(L + r).$$

Also,

$$\begin{aligned} d_S(x, y) &\geq d_S(u^m, h^k u^m) - d_S(u^m, x) - d_S(h^k u^m, y) \\ &\geq (|h^k|_S - 4m) - 4L(L + r) - 4L(L + r) \geq \frac{k}{L_1} - L_1 - 16L(L + r) \\ &\geq (nr + 16L(L + r)) - 16L(L + r) \geq nr. \end{aligned}$$

Thus, for each $\rho \in (0, 1]$

$$\sigma_\rho^n(r) \leq 2L_1L^2(L + r)(nr + 16L(L + r) + L_1 + 1) + 8L(L + r).$$

This implies that the subgroup divergence of H in A_Γ is at most quadratic. Therefore, the subgroup divergence of H in A_Γ is exactly quadratic. \square

6.2. Higher-degree polynomial subgroup divergence.

Definition 6.7. Let X be a geodesic space and x_0 one point in X . For each $\rho \in (0, 1]$, we define a function $\delta_\rho: [0, \infty) \rightarrow [0, \infty)$ as follows:

For each r , let $\delta_\rho(r) = \sup d_{\rho r}(x_1, x_2)$ where the supremum is taken over all $x_1, x_2 \in S_r(x_0)$ such that $d_{\rho r}(x_1, x_2) < \infty$.

The family of functions $\{\delta_\rho\}$ is the *divergence* of X with respect to the point x_0 , denoted Div_{X, x_0} .

In [Ger94], Gersten show that the divergence Div_{X, x_0} is, up to the relation \sim , a quasi-isometry invariant which is independent of the chosen base-point. The *divergence* of X , denoted Div_X , is then, up to the relation \sim , the divergence Div_{X, x_0} for some point x_0 in X .

If the space X has the *geodesic extension property* (i.e. any finite geodesic segment can be extended to an infinite geodesic ray), then it is not hard to show that $\delta_\rho \sim \delta_1$ for each $\rho \in (0, 1]$. In this case, we can consider the divergence of X as the function δ_1 .

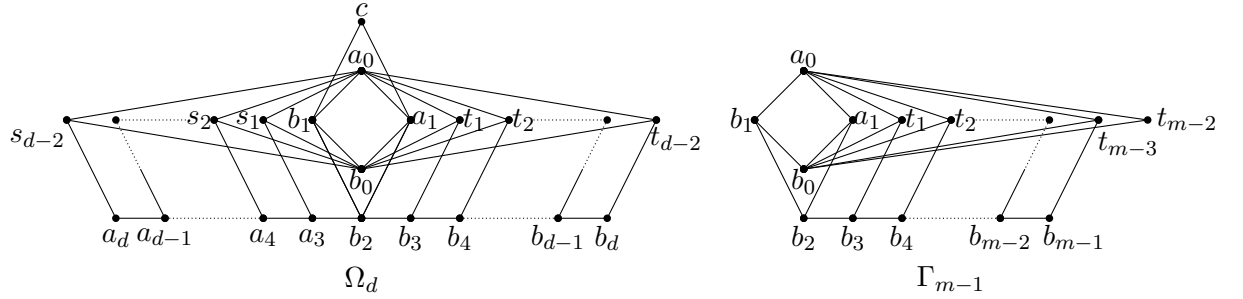


FIGURE 2

The *divergence* of a finitely generated group G , denoted $Div(G)$, is the divergence of its Cayley graphs.

The following definition was introduced by Levcovitz in [Lev] to study divergence in Coxeter groups.

Definition 6.8. Let Γ be a finite, connected, simplicial graph. A non-adjacent pair (s, t) is *rank 1* if s, t are not contained in some four-cycle of Γ . Additionally (s, t) are *rank n* if either every non-adjacent pair (s_1, s_2) , with $s_1, s_2 \in Lk(s)$, is rank $n - 1$ or every non-commuting pair (t_1, t_2) , with $t_1, t_2 \in Lk(t)$, is rank $n - 1$.

Proposition 6.9. Let Γ be a finite, connected, simplicial graph and n a positive integer. There is a polynomial f_n of degree n such that the following hold. Let (s, t) be a rank n pair of vertices of Γ and let H_1, H_2 be two hyperplanes of Σ_Γ of types s, t respectively such that their supports intersect. Let p be a vertex in the intersection between two support of H_1 and H_2 . The length of the any path from H_1 to H_2 which avoids the ball $B(p, r)$ is bounded below by $f_n(r)$.

The above proposition is a result from [Lev]. More precisely, two hyperplanes H_1 and H_2 are degree n M -separated in the sense of Definition 6.1 in [Lev] (see the proof of Theorem 7.9 in [Lev]). Therefore, there is a polynomial g_n of degree n such that the length of any path from H_1 to H_2 which avoids the ball $B(p, r)$ is bounded below by $g_n(r)$ (see Theorem 6.2 in [Lev]). Since the number of rank n pair of vertices in Γ is finite, we can choose a universal polynomial f_n of degree n as in the above lemma.

Theorem 6.10 (Theorem 7.9 in [Lev]). Let Γ be a finite, connected, simplicial graph. Suppose Γ contains a rank n pair (s, t) , then $Div(W)$ is bounded below by a polynomial of degree $n + 1$.

We now construct right-angled Coxeter groups with purely loxodromic subgroups of different subgroup divergence. More precisely, for each $d \geq 3$ let Ω_d be a graph in Figure 2. We will construct non-virtually cyclic purely loxodromic subgroups with subgroup divergence of polynomial of degree m

for $2 \leq m \leq d$. We remark that the graphs Γ_m in Image 2 were introduced by Dani-Thomas [DT15a] to study divergence of right-angled Coxeter groups and each graph Ω_d in Figure 2 is a variation of the graph Γ_d . We now prepare some lemmas and propositions that help for the construction of the desired purely loxodromic subgroups in G_{Ω_d} .

Lemma 6.11. *For each $d \geq 3$ let Ω_d be a graph as in Image 2. For each $3 \leq m \leq d$ all pairs (a_m, b_m) , (a_m, c) , (b_m, c) are rank $m - 1$.*

Proof. We first prove that for $3 \leq \ell \leq k \leq d$ each pair of non-adjacent vertices in $Lk(a_k)$ is rank $\ell - 2$. We will prove this by induction on ℓ . For $\ell = 3$ we observe that each pair of non-adjacent vertices in $Lk(a_k)$ ($k \geq 3$) are not contained in some four-cycle of Γ . Therefore, these pairs are all rank 1.

Assume that there is $4 \leq \ell_0 \leq d - 1$ such that each pair of non-adjacent vertices in $Lk(a_k)$ ($\ell_0 \leq k \leq d$) is rank $\ell_0 - 2$. We need to prove that each pair of non-adjacent vertices in $Lk(a_k)$ ($\ell_0 + 1 \leq k \leq d$) is rank $\ell_0 - 1$. We observe that $Lk(a_k) = \{s_{k-2}, a_{k-1}, a_{k+1}\}$ ($\ell_0 + 1 \leq k \leq d - 1$). By hypothesis induction, each pair of non-adjacent vertices in $Lk(a_{k-1})$, $Lk(a_{k+1})$ is rank $\ell_0 - 2$. Therefore, all pairs of non-adjacent vertices (a_{k-1}, s_{k-2}) , (a_{k+1}, s_{k-2}) , (a_{k-1}, a_{k+1}) are rank $\ell_0 - 1$. In other word, each pair of non-adjacent vertices in $Lk(a_k)$ ($\ell_0 + 1 \leq k \leq d - 1$) is rank $\ell_0 - 1$. For the case $k = d$ we see that $Lk(a_k) = Lk(a_d) = \{a_{d-1}, s_{d-2}\}$ and each pair of non-adjacent vertices in $Lk(a_{d-1})$ is rank $\ell_0 - 2$ by hypothesis induction. Therefore, pair of non-adjacent vertices in $Lk(a_d)$ is rank $\ell_0 - 1$ by hypothesis induction. Thus, for $3 \leq \ell \leq k \leq d$ each pair of non-adjacent vertices in $Lk(a_k)$ is rank $\ell - 2$. In particular, each pair of non-adjacent vertices in $Lk(a_m)$ ($3 \leq m \leq d$) is rank $m - 2$. By a similar argument, each pair of non-adjacent vertices in $Lk(b_m)$ ($3 \leq m \leq d$) is rank $m - 2$. This implies that for each $3 \leq m \leq d$ all pairs (a_m, b_m) , (a_m, c) , (b_m, c) are rank $m - 1$. \square

The following lemma is a directed result from Proposition 6.9 and Lemma 6.11.

Proposition 6.12. *For each $d \geq 3$ and $3 \leq m \leq d$ let Ω_d be the graph as in Image 2 and H_d^m the subgroup of G_{Ω_d} generated by c , a_m , and b_m . Then the subgroup divergence of H_d^m in G_{Ω_d} is bounded below by a polynomial of degree m .*

Proof. Let $\{\sigma_\rho^n\}$ be the subspace divergence of H_d^m in the Caley graph $\Sigma_{\Omega_d}^{(1)}$. Let f_{m-1} be the polynomials of degree $m - 1$ as in Proposition 6.9. We will prove that for each $n \geq 8$ and $\rho \in (0, 1]$

$$\sigma_\rho^n(r) \geq (r - 1)f_{m-1}(\rho r) \text{ for each } r > 1.$$

Let u and v be an arbitrary pair of points in $\partial N_r(H)$ such that $d_r(u, v) < \infty$ and $d_S(u, v) \geq nr$. Let γ be an arbitrary path that lies outside the ρr -neighborhood of H connecting u and v . We will prove that the length of γ is bounded below by $(r - 1)f_{m-1}(\rho r)$.

Let γ_1 be a geodesic of length r in $\Sigma_{\Omega_d}^{(1)}$ connecting u and some point x in H_d^m . Let γ_2 be another geodesic of length r in $\Sigma_{\Omega_d}^{(1)}$ connecting v and some point y in H_d^m . Let α be a geodesic in $\Sigma_{\Omega_d}^{(1)}$ connecting x and y . Obviously, each edge of α is labelled by a_m , b_m , or c . This implies that two hyperplanes determined by two different edges in α do not intersect. Since $d_S(x, y) \geq d_S(u, v) - 2r \geq (n-2)r \geq 6r$, there is a subpath β with length bounded below by r of α such that $\beta \cap (B(x, 2r) \cup B(y, 2r)) = \emptyset$. Also, the lengths of γ_1 and γ_2 are both r . This implies that each hyperplane determined by edge in β does not intersect $\gamma_1 \cup \gamma_2$. Therefore, each hyperplane determined by edge in β must intersect γ .

Assume that the path β is the concatenation of edges e_1, e_2, \dots, e_k , $k \geq r$ and let H_i be the hyperplane determined by edge e_i . Therefore, for each $i \in \{1, 2, \dots, k-1\}$ the $(i+1)^{th}$ vertex p_i of β lies in the intersection of the support of H_i and H_{i+1} . For each $i \in \{1, 2, \dots, k\}$ let x_i be a point in $H_i \cap \gamma$. Let γ_i be the subpath of γ connecting x_i and x_{i+1} for each $i \in \{1, 2, \dots, k-1\}$. Therefore, each γ_i is a path from H_i to H_{i+1} which avoids the ball $B(p_i, \rho r)$. Therefore, the length of each γ_i is bounded below by $f_{m-1}(\rho r)$ by Proposition 6.9 and Lemma 6.11. This implies that the length of γ is bounded below by $(k-1)f_{m-1}(\rho r)$. Also, $k \geq r$. Therefore, the length of γ is bounded below by $(r-1)f_{m-1}(\rho r)$. Thus, $\sigma_\rho^n(r) \geq (r-1)f_{m-1}(\rho r)$. Therefore, the subgroup divergence of H_d^m in G_{Ω_d} is bounded below by a polynomial of degree m . \square

Lemma 6.13. *For each $d \geq 3$ and $3 \leq m \leq d$ there is a polynomial g_{m-1} of degree $m-1$ such that the following holds. Let α be a geodesic ray based at e that is labelled by $a_1 b_1 a_1 b_1 \dots$. Let β be a geodesic ray based at e that is labelled by $b_{m-1} t_{m-2} b_{m-1} t_{m-2} \dots$. Then for each $r > 0$ there is a path outside $N_r(H_d^m)$ connecting $\alpha(r)$ and $\beta(r)$ with length bounded above by $g_{m-1}(r)$.*

Proof. Let Γ_{m-1} be a subgraph of Ω_d as in Image 2. Let S, S' be vertex sets of Ω_d, Γ_{m-1} respectively. Obviously, α and β be two geodesic rays in the 1-skeleton $\Sigma_{\Gamma_{m-1}}^{(1)}$ of the Davis complex $\Sigma_{\Gamma_{m-1}}$. Since the Cayley graph $\Sigma_{\Gamma_{m-1}}^{(1)}$ has geodesic extension property and the divergence of $\Sigma_{\Gamma_{m-1}}^{(1)}$ is a polynomial of degree $m-1$ (see Section 5 in [DT15a]), there is a polynomial g_{m-1} of degree $m-1$ such that for each $r > 0$ there is a path γ_r in $\Sigma_{\Gamma_{m-1}}^{(1)}$ with length bounded above by $g_{m-1}(r)$ connecting $\alpha(r)$, $\beta(r)$ and γ_r avoids the ball $B(e, r)$ in $\Sigma_{\Gamma_{m-1}}^{(1)}$. We now prove that each γ_r also lies outside $N_r(H_d^m)$ by showing that its vertices lie outside $N_r(H_d^m)$.

Let $\Phi: G_{\Omega_d} \rightarrow G_{\Gamma_{m-1}}$ be the group homomorphism induced by mapping each vertex of Γ_{m-1} to itself and each vertex outside Γ_{m-1} to e . This is not hard to check the following:

- (1) The map Φ is a well-defined group homomorphism.
- (2) $\Phi(u) = u$ for each u in $G_{\Gamma_{m-1}}$ and $\Phi(h) = e$ for each h in H_d^m .

(3) $|\Phi(g)|_{S'} \leq |g|_S$ for each g in G_{Ω_d} .

For each vertex u in γ_r , u is a group element in $G_{\Gamma_{m-1}}$ with $|u|_{S'} \geq r$. Assume that $m = d_S(u, H_d^m)$. Then there is h in H_d^m such that $m = d_S(h, u) = |h^{-1}u|_S$. Therefore,

$$m = |h^{-1}u|_S \geq |\Phi(h^{-1}u)|_{S'} = |u|_{S'} \geq r.$$

This implies that each vertex in γ_r lies outside $N_r(H_d^m)$. Therefore, each path γ_r also lie outside $N_r(H_d^m)$. \square

Proposition 6.14. *For each $d \geq 3$ and $3 \leq m \leq d$ let Ω_d be the graph as in Image 2 and H_d^m the subgroup of G_{Ω_d} generated by c , a_m , and b_m . Then the subgroup divergence of H_d^m in G_{Ω_d} is bounded above by a polynomial of degree m .*

Proof. Let α, β be geodesic rays as in Lemma 6.13 and g_{m-1} a polynomial of degree $m-1$ as in this lemma. Let $\{\sigma_\rho^n\}$ be the subspace divergence of H_d^m in the Cayley graph $\Sigma_{\Omega_d}^{(1)}$. We will prove that function $\sigma_\rho^n(r)$ is bounded above by some quadratic function for each $n \geq 2$ and $\rho \in (0, 1]$.

For each $r > 1$ there is a path γ_r outside $N_r(H_d^m)$ connecting $\alpha(r)$ and $\beta(r)$ with length bounded above by $g_{m-1}(r)$. Since the generator b_m commutes with all edge labels of β , two points $\beta(r)$ and $b_m\beta(r)$ lie on the boundary of a 2-cell in Σ_{Ω_d} . Therefore, there is a path α_1 outside $N_r(H_d^m)$ connecting $\beta(r)$ and $b_m\beta(r)$ with length bounded above by 3. Similarly, the generator c commutes with all edge labels of α , two points $b_m\alpha(r)$ and $(b_mc)\alpha(r)$ lie on the boundary of a 2-cell in Σ_{Ω_d} . Therefore, there is a path α_2 outside $N_r(H_d^m)$ connecting $b_m\alpha(r)$ and $(b_mc)\alpha(r)$ with length bounded above by 3. Also $b_m\gamma_r$ is a path outside $N_r(H_d^m)$ connecting $b_m\alpha(r)$ and $b_m\beta(r)$ with length bounded above by $g_{m-1}(r)$. Therefore, $\eta_1 = \gamma_r \cup \alpha_1 \cup b_m\gamma_r \cup \alpha_2$ is a path outside $N_r(H_d^m)$ connecting $\alpha(r)$ and $(b_mc)\alpha(r)$ with length bounded above by $2g_{m-1}(r) + 6$.

For each $n \geq 2$, let k be an integer between nr and $2nr$. Let

$$\eta = \eta_1 \cup (b_mc)\eta_1 \cup (b_mc)^2\eta_1 \cup \dots \cup (b_mc)^{k-1}\eta_1.$$

Then, η is a path outside $N_r(H_d^m)$ connecting $\alpha(r)$ and $(b_mc)^k\alpha(r)$ with length bounded above by $k(2g_{m-1}(r) + 6)$. Therefore,

$$d_{\rho r}(\alpha(r), (b_mc)^k\alpha(r)) \leq k(2g_{m-1}(r) + 6) \leq 2nr(2g_{m-1}(r) + 6)$$

Also,

$$d_S(\alpha(r), (b_mc)^k\alpha(r)) \geq d_S(e, (b_mc)^k) - 2r \geq 2k - 2r \geq (2n - 2)r \geq nr.$$

Therefore, $\sigma_\rho^n(r) \leq 2nr(2g_{m-1}(r) + 6)$. This implies that the subgroup divergence of H in G_{Ω_d} is bounded above by a polynomial of degree m . \square

By using similar techniques as in Lemma 6.13, Proposition 6.14 and applying Theorem 6.2, we also obtain the following proposition.

Proposition 6.15. *For each $d \geq 3$ let Ω_d be the graph as in Image 2 and H_d^2 the subgroup of G_{Ω_d} generated by c , s_1 , and t_1 . Then the subgroup divergence of H_d^2 in G_{Ω_d} is exactly a quadratic function.*

Theorem 6.16. *For each $d \geq 3$ let Ω_d be the graph as in Image 2. Let H_d^2 be the subgroup of G_{Ω_d} generated by the set $\{c, s_1, t_1\}$. For each $3 \leq m \leq d$ let H_d^m the subgroup of G_{Ω_d} generated by the set $\{c, a_m, b_m\}$. Then for each $2 \leq m \leq d$ subgroup H_d^m is a purely loxodromic subgroup of G_{Ω_d} , H_d^m is isomorphic to the group $F = \langle s, t, u \mid s^2 = t^2 = u^2 = e \rangle$, and the subgroup divergence of H_d^m in G_{Ω_d} is a polynomial of degree m .*

Proof. We first consider the case $3 \leq m \leq d$. It is not hard to see that each infinite order element h in H_d^m can be written as a reduced word $s_1 s_2 \cdots s_m$, where each s_i belongs to the set $\{a_m, b_m, c\}$ and s_i, s_{i+1} are two different elements in $\{a_m, b_m, c\}$. Therefore, H_d^m is a join-busting subgroup. This implies that H_d^m is a purely loxodromic subgroup. By Propositions 6.12 and 6.14, the subgroup divergence of H_d^m in G_{Ω_d} is a polynomial of degree m . By a similar argument the subgroup H_d^2 is also join-busting. Therefore, H_d^2 is also a purely loxodromic subgroup. The fact that the subgroup divergence of H_d^2 in G_{Ω_d} is a quadratic function can be seen from Proposition 6.15. It is also obvious that all special subgroups are isomorphic to the group $F = \langle s, t, u \mid s^2 = t^2 = u^2 = e \rangle$. \square

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